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ON A CLASS OF AUTOMORPHIC FUNCTIONS*

BY

J. I. HUTCHINSON

In a paper entitled *Ueber die Darstellung einiger Fälle der automorphen Primformen durch specielle Thetareihen*,† BURKHARDT has considered the monodromy group of the Riemann surface

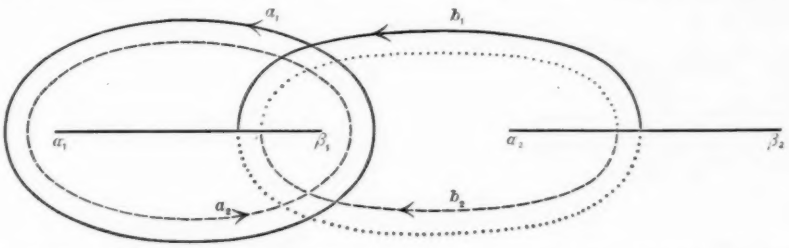
$$y^3 = (x - a_1)(x - a_2)(x - \beta_1)^2(x - \beta_2)^2, \tag{1}$$

and has shown how a certain prime form which is automorphic for the group can be expressed by a theta series.

In the following article further results are given concerning this group and the functions belonging to it, the chief object in view being to obtain explicit analytic formulæ by means of which all functions of the group can be represented. To this end the theta-fuchsian functions of POINCARÉ are introduced and their expressions in terms of the hyperelliptic theta series deduced. It is then found that every function which is automorphic for the group considered can be expressed rationally in terms of the quotient of two particular theta-fuchsian functions.

§ 1. *The Riemann surface.*

The Riemann surface defined by (1) is of genus $p = 2$, and depends on one arbitrary constant. It is convenient to choose the two pairs of cross-cuts as indicated



in the figure, where an unbroken line is used to represent a path in the first sheet, a broken line one in the second sheet and a dotted line one in the third sheet.

* Presented to the Society at the Ithaca meeting, August 19, 1901. Received for publication November 2, 1901.

† Mathematische Annalen, vol. 42 (1893), p. 185. See also articles by HURWITZ and RAUSENBERGER, in vol. 20 (1882), pp. 47, 125, 201.

Two linearly independent integrals are

$$u_1 = \int \frac{dx}{y}, \quad u_2 = \int \frac{(x-\beta_1)(x-\beta_2)\,dx}{y^2}.$$

If for a particular value of x the corresponding values of y in the first, second, and third sheets of the surface are denoted by y_1, y_2, y_3 respectively, we shall assume $y_2 = \rho y_1$, $y_3 = \rho^2 y_2$, where $\rho = \frac{1}{2}(-1+\sqrt{-3})$.

Denoting the moduli of periodicity of the integrals u_ν at the cuts a_1, a_2, b_1, b_2 by $A_{\nu 1}$, $A_{\nu 2}$, $B_{\nu 1}$, $B_{\nu 2}$ respectively, one sees easily from the above figure that

$$A_{\nu 2} = -\rho^\nu A_{\nu 1}, \quad B_{\nu 2} = \rho^{2\nu} B_{\nu 1},$$

and derives the following table of periods.

	a_1	a_2	b_1	b_2
u_1	A_{11}	$-\rho A_{11}$	B_{11}	$\rho^2 B_{11}$
u_2	$-\rho A_{22}$	A_{22}	$\rho^2 B_{22}$	B_{22}

On account of the bilinear relations among the periods, these are subject to the condition,

$$A_{11}B_{22} - A_{22}B_{11} = 0.$$

The table of moduli for the normal integrals v_1, v_2 then assumes the form,

(2)

	a_1	a_2	b_1	b_2
v_1	πi	0	$\frac{2\pi i z}{\sqrt{3}}$	$-\frac{\pi i z}{\sqrt{3}}$
v_2	0	πi	$-\frac{\pi i z}{\sqrt{3}}$	$\frac{2\pi i z}{\sqrt{3}}$

where

$$z = i\rho^2 \frac{B_{11}}{A_{11}}.$$

If we write $x + iy$ for z, the condition of inequality among the moduli is

(3) $$y > 0.$$

§ 2. *Generation and properties of the group G.*

We now proceed to consider the effect on the moduli of a monodromy of the branch points. First, let a_1 move positively around β_1. Suppose that corresponding to this change the Riemann surface and its cross-cuts undergo a con-

tinuous deformation. Using accents to distinguish the moduli of periodicity along the deformed cuts we have

$$A'_{11} = A_{11} + (\rho - 1) B_{11},$$

$$B'_{11} = B_{11},$$

whence

$$(\tau) \qquad z' = \frac{z}{1 - \sqrt{3}z}.$$

In the second place interchange a_1 and a_2. This gives rise to the transformation

$$A'_{11} = -B_{11}, \quad B'_{11} = -\rho^2 A_{11},$$

which is equivalent to

$$(T) \qquad z' = -\frac{1}{z}.$$

It is easy to verify that any other monodromy of the branch points leads to a transformation which can be compounded of the two already obtained.

These two transformations generate a group of infinite order which will be denoted by G. It will be convenient to replace τ as a generating operation of the group by the substitution

$$S = T\tau T, \quad \text{that is,} \quad z' = z + \sqrt{3}.$$

The transformation S divides the z-plane into an infinite number of strips parallel to the y-axis and of width $\sqrt{3}$, while T interchanges the outside and inside of the unit circle. Accordingly we may take as a fundamental region for the group G a strip in the positive half plane bounded by the lines $x = \pm \frac{1}{2}\sqrt{3}$, and the unit circle.

Representing each transformation by the determinant of its coefficients, we have the following theorem.

Every operation of the group is of one of the two types:

$$(A) \quad \begin{vmatrix} a & \beta\xi \\ \gamma\xi & \delta \end{vmatrix}, \qquad (B) \quad \begin{vmatrix} a\xi & b \\ c & d\xi \end{vmatrix},$$

where $\xi = \sqrt{3}$, *and* $a, b, \cdots, \alpha, \beta, \cdots$ *are integers satisfying the relations,*

$$(4) \qquad \begin{aligned} 3ad - bc &= 1, \\ a\delta - 3\beta\gamma &= 1. \end{aligned}$$

For, S is of type (A), and T is of type (B). Also, the product of two sub-

stitutions of the same type gives one of type (A) while the product of two of different types gives one of type (B).

Conversely, every operation of type (A), *or of type* (B), *is contained in* G.

For, starting with (A), if β is not positive and less than $|\delta|$, we apply S^m and obtain

$$\begin{vmatrix} a+3m\gamma & (\beta+m\delta)\xi \\ \gamma\xi & \delta \end{vmatrix} = \begin{vmatrix} a' & \beta'\xi \\ \gamma\xi & \delta \end{vmatrix}.$$

By properly choosing m we can make $0<\beta'<|\delta|$ since, on account of (4), β and δ are relatively prime, and likewise β' and δ.

Next, applying

$$TS^nT = \begin{vmatrix} -1 & 0 \\ n\xi & -1 \end{vmatrix},$$

we have

$$\begin{vmatrix} -a' & -\beta'\xi \\ (a'n-\gamma)\xi & 3n\beta'-\delta \end{vmatrix} = \begin{vmatrix} -a' & -\beta'\xi \\ \gamma'\xi & \delta' \end{vmatrix}.$$

By a proper choice of n, δ' can be made positive and less than $3\beta'$.

There are now three possibilities to consider:

1. $0<\delta'<\beta'$,
2. $\beta'<\delta'<2\beta'$,
3. $2\beta'<\delta'<3\beta'$.

In case 1 we proceed as at first.

In case 2 by applying S we obtain $\beta''=\delta'-\beta'$, which satisfies the condition

$$0<\beta''<\beta'<\delta'.$$

But since

$$\delta'+\delta' = 2\beta'+2\beta'' \quad \text{and} \quad \delta'<2\beta',$$

it follows that

$$\delta'>2\beta''.$$

If at the same time $\delta'<3\beta''$, case 3 is to be applied, otherwise we multiply by some power of TST as before.

In case 3 by applying $TS^{-1}T$ we get $\delta''(=3\beta'-\delta')$ with the condition

$$0<\delta''<\beta'.$$

We can continue in this manner to reduce the numerical value of the β-coefficient until it reaches zero. The resulting substitution is a power of TST.

Starting with an operation of type (B) we first apply T. The result being then of type (A) we proceed as above.

§ 3. *Transformation of the hyperelliptic thetas.*

In order to obtain functions which are automorphic for the group G, we determine in the first place the effect of S and T upon the theta constants, that is, the theta functions with zero arguments and moduli a_{jk}, where

$$a_{jj} = \frac{2\pi i z}{\sqrt{3}}, \quad a_{12} = a_{21} = -\frac{\pi i z}{\sqrt{3}}.$$

For this purpose we make use of the transformation theory of the theta functions. Let b_{jk} be the new, or transformed moduli. Then for the substitution S,

$$b_{jj} = \frac{2\pi i z'}{\sqrt{3}}, \quad b_{12} = -\frac{\pi i z'}{\sqrt{3}},$$

where

$$z' = z + \sqrt{3}.$$

The transformation formulæ for the ϑ-functions require us to determine integral coefficients * α_{jk}, β_{jk}, γ_{jk}, δ_{jk} satisfying the identities

$$\sum_{\kappa} b_{\nu\kappa}[\alpha_{\kappa\mu}\pi i + \sum_{\lambda} \beta_{\kappa\lambda}\alpha_{\mu\lambda}] = \pi i\,[\gamma_{\nu\mu}\pi i + \sum_{\lambda} \delta_{\nu\lambda}\alpha_{\mu\lambda}],$$

and subject also to certain bilinear relations. Substitute in this formula for $b_{\nu\kappa}$ and $a_{\mu\lambda}$ in terms of z, and equate coefficients of like powers of z in both members. The final result is that

$$\left|\begin{array}{cc|cc} \alpha_{11} & \alpha_{12} & \beta_{11} & \beta_{12} \\ \alpha_{21} & \alpha_{22} & \beta_{21} & \beta_{22} \\ \hline \gamma_{11} & \gamma_{12} & \delta_{11} & \delta_{12} \\ \gamma_{21} & \gamma_{22} & \delta_{21} & \delta_{22} \end{array}\right| = \left|\begin{array}{cc|cc} m & n & \multicolumn{2}{c}{\multirow{2}{*}{0}} \\ -n & m+n & & \\ \hline 2m+n & n-m & m+n & n \\ -m-2n & 2m+n & -n & m \end{array}\right|,$$

or

$$= \left|\begin{array}{cc|cc} -m & m+n & \multicolumn{2}{c}{\multirow{2}{*}{0}} \\ n & m & & \\ \hline -2m-n & m+2n & -m & n \\ m+2n & m-n & m+n & m \end{array}\right|,$$

where m, n are integers subject to the condition

$$m^2 + mn + n^2 = 1.$$

Let $\vartheta\left[\begin{smallmatrix} g \\ h \end{smallmatrix}\right](z)$ denote a ϑ-function of the first order with characteristic $\left[\begin{smallmatrix} g \\ h \end{smallmatrix}\right]$,

* We use here the notation of KRAZER and PRYM, *Neue Grundlagen*, etc., p. 121.

arguments zero, and moduli a_{jk}; also let $\vartheta\left[\begin{smallmatrix}g\\h\end{smallmatrix}\right](z')$ be a similar function with moduli b_{jk}. Then one of the above transformations leads to the formula

$$\text{(I)} \qquad \vartheta\left[\begin{smallmatrix}g\\h\end{smallmatrix}\right](z') = e^{-2\pi i(g_1^2 - g_1g_2 + g_2^2)}\,\vartheta\left[\begin{smallmatrix}g_1 & g_2\\ h_1+2g_1-g_2 & h_2+2g_2-g_1\end{smallmatrix}\right](z).$$

Corresponding to T we find in a similar manner the transformation

$$\left|\begin{array}{cc|cc} & 0 & -m & -m-n \\ & & n & -m \\ \hline m & n & & \\ -m-n & m & \multicolumn{2}{c}{0} \end{array}\right|,$$

or

$$\left|\begin{array}{cc|cc} & 0 & -m-n & -n \\ & & -n & m \\ \hline m & n & & \\ n & -m-n & \multicolumn{2}{c}{0} \end{array}\right|,$$

and the resulting ϑ-formula

$$\text{(II)} \qquad \vartheta\left[\begin{smallmatrix}g\\h\end{smallmatrix}\right](z') = -\,ize^{2\pi i \Sigma g_j h_j}\,\vartheta\left[\begin{smallmatrix}h_2 & -h_1\\ -g_2 & g_1\end{smallmatrix}\right](z).$$

It is to be remarked that (I) and (II) were obtained by making a particular choice for m and n. Another choice for these numbers leads to different results. This indicates the existence of special relations among the ϑ-constants which do not occur in the general case. These are conveniently obtained by considering the transformation for which $b_{jk} = a_{jk}$.

Computing the coefficients in the manner above explained we arrive at the result

$$\left|\begin{array}{cc|cc} m & n & & \\ -n & m+n & \multicolumn{2}{c}{0} \\ \hline & & m+n & n \\ \multicolumn{2}{c|}{0} & -n & m \end{array}\right|, \quad \text{or} \quad \left|\begin{array}{cc|cc} m & n & & \\ m+n & -m & \multicolumn{2}{c}{0} \\ \hline & & m & m+n \\ \multicolumn{2}{c|}{0} & n & -m, \end{array}\right|,$$

which leads to only two distinct relations, viz.,

$$\text{(III)} \qquad \begin{aligned} \vartheta\left[\begin{smallmatrix}g\\h\end{smallmatrix}\right](z) &= \vartheta\left[\begin{smallmatrix}g_2 & g_1\\ h_2 & h_1\end{smallmatrix}\right](z), \\ \vartheta\left[\begin{smallmatrix}g\\h\end{smallmatrix}\right](z) &= \vartheta\left[\begin{smallmatrix}g_1 & g_1-g_2\\ h_1+h_2 & -h_2\end{smallmatrix}\right](z). \end{aligned}$$

These formulæ reduce the 10 ϑ-constants with half integer characteristics to the four,*

$$\vartheta\begin{bmatrix}00\\00\end{bmatrix}(z), \quad \vartheta\begin{bmatrix}11\\00\end{bmatrix}(0), \quad \vartheta\begin{bmatrix}00\\11\end{bmatrix}(z), \quad \vartheta\begin{bmatrix}11\\11\end{bmatrix}(z),$$

which will be denoted by

$$\vartheta, \quad \vartheta_1, \quad \vartheta_2, \quad \vartheta_3, \tag{5}$$

respectively. These four functions satisfy the identical relations,

$$\begin{aligned} \vartheta^2 + \vartheta_3^2 &= \vartheta_1^2 + \vartheta_2^2, \\ \vartheta^4 - \vartheta_3^4 &= \vartheta_1^4 + \vartheta_2^4, \end{aligned} \tag{6}$$

which are readily deducible from the general relations among the thetas.

The effect of S and T on the functions (5) is shown in the following table.

(7)

	ϑ	ϑ_1	ϑ_2	ϑ_3
S	ϑ	$-i\vartheta_3$	ϑ_2	$-i\vartheta_1$
T	$-iz\vartheta$	$-iz\vartheta_2$	$-iz\vartheta_1$	$-iz\vartheta_3$

Using κ, λ, μ (instead of the usual κ^2, λ^2, μ^2) to denote the Rosenhain moduli we have, on account of (III),

$$\kappa = \frac{\vartheta_1^2}{\vartheta^2}, \quad \lambda = \frac{\vartheta_3^2}{\vartheta_2^2}, \quad \mu = \frac{\vartheta_1^2\vartheta_3^2}{\vartheta^2\vartheta_2^2}, \tag{8}$$

whence

$$\kappa\lambda = \mu = \frac{\lambda^2}{1 - \lambda + \lambda^2}, \tag{9}$$

the second equality being deducible from (6).

§ 4. *The theta-fuchsian functions.*

We have now to show that every automorphic function belonging to G can be expressed rationally in terms of any one of the Rosenhain moduli.

For this purpose we observe that the group G is of genus zero, and hence it follows that every function belonging to G can be expressed rationally in terms of any one which takes a given value once, and but once, in the fundamental region. In order to determine such functions, it is convenient to consider in this connection the theta-fuchsian functions introduced by POINCARÉ.

A theta-fuchsian function $\Theta(z)$ is defined by the relation

$$\Theta(z') = (\gamma z + \delta)^{2m}\Theta(z),$$

* In conformity to the customary notation the denominators of the characteristics are omitted.

where $$z' = \frac{\alpha z + \beta}{\gamma z + \delta}$$

is a substitution of the given group, and m is a positive integer called the degree of the group. It is necessary to add that $\Theta(z)$ is one-valued, and in what follows we shall suppose that it has no discontinuities except on the x-axis.*

We have first of all to determine the number of zeros of $\Theta(z)$ in the fundamental region R. The two points $a = \frac{1}{2}(i - \sqrt{3})$, $a' = \frac{1}{2}(i + \sqrt{3})$ are vertices of R, the first of which will be regarded as included in the fundamental region, and the other excluded from it. Starting now with a, we integrate $d \log \Theta(z)$ positively along the boundary of R, and along circular arcs of infinitesimal radius excluding the regions in the vicinity of the vertices a, a', $b(=i)$, $c(=i\infty)$. The circular arc excluding c is the straight line drawn from a point z_1, on the line $x = \frac{1}{2}\sqrt{3}$, parallel to the x-axis and ending in the point $z_1' = z_1 - \sqrt{3}$. The imaginary part of z_1 is regarded as indefinitely great.

Since $\Theta(z + \sqrt{3}) = \Theta(z)$, we assume that $\Theta(z)$ can be expanded into a series of the form

$$\Theta(z) = \sum_{\kappa=n}^{\kappa=\infty} A_\kappa t^\kappa,$$

where

$$t^\kappa = e^{\frac{2\pi i z}{\sqrt{3}}},$$

and n is a finite integer. If n is negative I shall say that $\Theta(c)$ is infinite of the n-th order; and, on the other hand, that it is zero of like order if n is positive.

The line from z_1 to z_1' corresponds to the circumference of a circle about the origin in the t-plane, and generated by negative rotation from π to $-\pi$. Hence

$$\int_{z_1}^{z_1'} d \log \Theta(z) = -2n\pi i.$$

Suppose in the next place that $z = a$ is a vertex of the fundamental region (or a congruent point) not on the x-axis, and let p denote the period of the substitution having a for fixed point. According to a theorem of POINCARÉ's,† if

$$m \equiv -q \pmod{p}, \quad 0 \leqq q < p,$$

$\Theta(z)$ can be expanded into a series of ascending powers of $z - a$ of the form

$$A_q(z-a)^q + A_{q+p}(z-a)^{q+p} + \cdots,$$

so that $z = a$ is a zero of order q for every function of the same degree m.

I wish at this point to distinguish between *fixed* and *movable* zeros of $\Theta(z)$. A fixed zero will be one which is common to all functions of the same degree m.

* Under this restriction we have what POINCARÉ calls a function of the second species. In the present case a Θ-function having poles is the product of an automorphic function and a Θ-function of the second species, so that the above assumption is no restriction of generality.

† Acta Mathematica, vol. 1 (1882), p. 218.

All others will be called movable zeros, since they change their positions in the z-plane when we pass from one Θ-function to another.

In accordance with this distinction the point a is a fixed zero of order q. If one, or more, of the coefficients $A_q, \cdots$ should vanish, we say that a corresponding number of movable zeros has coincided with the fixed zero, and that a then counts as a fixed zero of order q, and a movable zero of order equal to the number of coefficients which have vanished.

On the supposition that $\Theta(c)$ is infinite of order n, we obtain for the number M of movable zeros situated in the fundamental region R the formula

$$M = n + \frac{m}{3} - \frac{q_a}{6} - \frac{q_b}{2},$$

where q_a and q_b are the values of q at the two vertices a and b respectively.

If, on the other hand, $\Theta(z)$ becomes zero of order n at c, we obtain for M the value

$$M = \frac{m}{3} - \frac{q_a}{6} - \frac{q_b}{2}, \tag{10}$$

counting the zero at c as a movable zero of order n.

Restricting ourselves to the case in which $\Theta(z)$ is everywhere finite in R (including $z = c$) we observe that Θ-functions of degree $m = 1$ do not exist, since for this value of m formula (10) gives $M = -1$. Further, $M = 0$ when $m = 2$, or 3, and hence only one function of degree 2, or 3 can exist. We see by reference to the table (7) that $\vartheta^4(z)$ is the Θ-function of degree 2.

Since $q_a = 4$, $q_b = 0$ when $m = 2$, it follows that $\vartheta^4(z)$ has a fixed zero at a (and all congruent points) of order 4; or, $\vartheta(z)$ is zero of the first order at these points, and nowhere else.

For $m = 4$, $M = 1$, we have two linearly independent functions,

$$\Theta_1(z) = \vartheta^8, \quad \Theta_2(z) = \vartheta^2 \vartheta_1^2 \vartheta_2^2 \vartheta_3^2.$$

Every other Θ-function of the same degree is expressible linearly in terms of these two. The quotient

$$\frac{\Theta_1(z)}{\Theta_2(z)} = \eta(z)$$

is an automorphic function which vanishes, and becomes infinite only once in the fundamental region, and hence takes any given value but once in the same region. Accordingly every function of the group can be expressed rationally in terms of $\eta(z)$. From (6) and (8) it readily follows that

$$\eta(z) = \frac{\lambda^2(1-\lambda)^2}{(1-\lambda+\lambda^2)^3} = \kappa^2(1-\kappa) = \mu(1-\mu)^2.$$

§ 5. *Expression of the Rosenhain moduli in terms of the anharmonic ratio of the branch points.*

It now remains to express the Rosenhain moduli in terms of $a_1, a_2, \beta_1, \beta_2$. This is accomplished by transforming the given hyperelliptic integrals into the Rosenhain normal form by means of a birational transformation. This requires the introduction of a new independent variable which is two-valued on the given Riemann surface. Such is the function

$$\xi = \sqrt[3]{\frac{(x-a_1)(x-a_2)}{(x-\beta_1)(x-\beta_2)}} = \frac{y}{(x-\beta_1)(x-\beta_2)}.$$

Differentiating, we obtain

$$3\xi^2 d\xi = \frac{R dx}{(x-\beta_1)^2(x-\beta_2)^2},$$

where

$$R = (a_2-\beta_2)(x-a_1)(x-\beta_1) + (a_1-\beta_1)(x-a_2)(x-\beta_2).$$

Hence

$$\frac{3(x-\beta_1)(x-\beta_2)}{R}\, d\xi = \frac{(x-\beta_1)(x-\beta_2)}{y^2}\, dx.$$

Writing η in the place of $\dfrac{R}{(x-\beta_1)(x-\beta_2)}$ and integrating, we have *

$$3\int \frac{d\xi}{\eta} = u_1.$$

Similarly

$$3\int \frac{\xi\, d\xi}{\eta} = u_2.$$

The expression of η in terms of ξ gives the relation

$$\eta^2 = (\beta_1-\beta_2)^2\xi^6 + 2\left[(a_1-\beta_1)(a_2-\beta_2) + (a_1-\beta_2)(a_2-\beta_1)\right]\xi^3 + (a_1-a_2)^2.$$

We find the branch points of the new surface by solving the equation $\eta = 0$. Denoting these by $\xi_i (i = 1, 2, \cdots, 6)$, and writing

$$A = (a_1-\beta_1)(a_2-\beta_2) + (a_1-\beta_2)(a_2-\beta_1),$$

$$\Delta = A^2 - (a_1-a_2)^2(\beta_1-\beta_2)^2 = 4(a_1-\beta_1)(a_2-\beta_2)(a_1-\beta_2)(a_2-\beta_1),$$

$$L = \left[\frac{-A+\sqrt{\Delta}}{(\beta_1-\beta_2)^2}\right]^{\frac{1}{3}}, \quad M = \left[\frac{-A-\sqrt{\Delta}}{(\beta_1-\beta_2)^2}\right]^{\frac{1}{3}},$$

* Cf. BURKHARDT, l. c., p. 209. BURKHARDT remarks that the hyperelliptic field is reducible to the elliptic by a transformation of higher degree. The transformation, which is of degree 2, is given on page 14 of my Chicago dissertation, *On the reduction of hyperelliptic functions etc.*, Göttingen, 1897.

we have $$\xi_1 = L, \quad \xi_2 = \rho L, \quad \xi_3 = \rho^2 L,$$
$$\xi_4 = M, \quad \xi_5 = \rho M, \quad \xi_6 = \rho^2 M.$$

If ω represents the anharmonic ratio

$$[\beta_1 a_1 \beta_2 a_2] = \frac{(a_1 - \beta_1)(a_2 - \beta_2)}{(a_1 - \beta_2)(a_2 - \beta_1)},$$

then

$$\frac{M}{L} = \left[\frac{1 + \sqrt{\omega}}{1 - \sqrt{\omega}}\right]^2,$$

and the expressions for the ROSENHAIN moduli take the form:

$$\kappa = [\xi_1 \xi_5 \xi_4 \xi_2] = \frac{L^2 + LM + M^2}{3LM},$$

$$\lambda = [\xi_1 \xi_3 \xi_4 \xi_2] = \frac{M - \rho L}{L - \rho M},$$

$$\mu = [\xi_1 \xi_6 \xi_4 \xi_2] = \frac{(L - \rho^2 M)(M - \rho L)}{3LM}.$$

It is easy to verify that these expressions satisfy (9).

From the formulæ just deduced we easily find

$$-\eta(z) = \kappa^2(\kappa - 1) = \frac{16}{27}\frac{\omega}{(1 - \omega)^2}. \tag{11}$$

Hence, *every automorphic function belonging to the group G is rationally expressible in terms of* ω, *i. e., in terms of the anharmonic ratio of the four branch points* $a_1, a_2, \beta_1, \beta_2$. On the other hand it follows from the particular form of the expression in ω in (11) that *every rational function of* $\omega + \omega^{-1}$ *is an automorphic function of the group.*

CORNELL UNIVERSITY, *August*, 1901.

CONCERNING THE EXISTENCE OF SURFACES CAPABLE OF CONFORMAL REPRESENTATION UPON THE PLANE IN SUCH A MANNER THAT GEODETIC LINES ARE REPRESENTED BY A PRESCRIBED SYSTEM OF CURVES*

BY

HENRY FREEMAN STECKER

Introduction.—This paper is in continuation of a previous paper† under nearly the same title. The notation given there is used in this paper with the exception that u, v are here used instead of μ, ν.

We are concerned with a doubly infinite system of *given* curves:

$$f_3(u, v) + Af_2(u, v) + Bf_1(u, v) = 0, \tag{1}$$

of which the differential equation is†

$$a_1du^3 + a_4dv^3 + a_2du^2\,dv + a_3du\,dv^2 + a_5(du\,d^2v - dv\,d^2u) = 0. \tag{2}$$

The geodetic lines of any surface are given by:

$$\begin{aligned}&(EF_u - \tfrac{1}{2}EE_v - \tfrac{1}{2}FE_u)du^3 + (-GF_v + \tfrac{1}{2}GG_u + \tfrac{1}{2}FG_v)dv^3\\ &+ (EG_u - \tfrac{3}{2}FE_v - \tfrac{1}{2}GE_u + FF_u)du^2\,dv\\ &+ (-GE_v + \tfrac{3}{2}FG_u + \tfrac{1}{2}EG_v - FF_v)du\,dv^2\\ &+ (EG - F^2)(du\,d^2v - dv\,d^2u) = 0 \qquad (F_u = \partial F/\partial u, \text{ etc.}),\end{aligned} \tag{3}$$

where u and v are Gaussian coördinates on the surface. A comparison of (2) and (3) leads to the following system of partial differential equations:

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† Transactions of the American Mathematical Society, vol. 2, p. 152.

$$
(4)\quad
\begin{aligned}
&(a) && EF_u - \tfrac{1}{2}EE_v - \tfrac{1}{2}FE_u = \frac{a_1}{a_5}(EG - F^2),\\
&(b) && -GF_v + \tfrac{1}{2}GG_u + \tfrac{1}{2}FG_v = \frac{a_4}{a_5}(EG - F^2),\\
&(c) && EG_u - \tfrac{3}{2}FE_v - \tfrac{1}{2}GE_u + FF_u = \frac{a_2}{a_5}(EG - F^2),\\
&(d) && -GE_v + \tfrac{3}{2}FG_u + \tfrac{1}{2}EG_v - FF_v = \frac{a_3}{a_5}(EG - F^2).
\end{aligned}
$$

The solution of the problem depends upon that of this system.

Multiply equation (a) by $-3F/E$ and add to equation (c). The result is:

$$
(5)\quad -2FF_u + EG_u - \tfrac{1}{2}GE_u + \tfrac{3}{2}\frac{F^2}{E}E_u = \left(\frac{a_2}{a_5} - 3\frac{a_1}{a_5}\frac{F}{E}\right)(EG - F^2).
$$

Dividing this equation through by $EG - F^2$, integrating with respect to u, and representing by $\psi(v)$ an arbitrary function of v only, we have:

$$
(6)\quad \frac{EG - F^2}{E^3} = \psi(v)\, e^{\int \frac{a_2}{a_5} du}\, e^{-3\int \frac{a_1}{a_5}\frac{F}{E} du}.
$$

In like manner, multiplying equation (b) by $-3F/G$, and adding to equation (d), we find

$$
(7)\quad \frac{G^3}{EG - F^2} = \phi(u)\, e^{\int \frac{a_3}{a_5} dv}\, e^{-3\int \frac{a_4}{a_5}\frac{F}{G} dv},
$$

where $\phi(u)$ is an arbitrary function of u only.

Beltrami's investigations* were for the case, $a_1 = a_2 = a_3 = a_4 = 0$, that is, where both of the exponentials in the right hand members of (6) and (7) become unity.

In my previous paper I considered the case $F = 0$, that is, where (6) and (7) assume the forms:

$$
(8)\quad \frac{EG - F^2}{E^3} = \psi(v)\, e^{\int \frac{a_2}{a_5} du},
$$

$$
(9)\quad \frac{G^3}{EG - F^2} = \phi(u)\, e^{\int \frac{a_3}{a_5} dv}.
$$

It is proposed to investigate the case in which $a_1 = a_4 = 0$, while a_2 and a_3 are unrestricted.

Under these restrictions, the system (6) and (7) assumes the form (8) and (9), that is, the same form as for the case $F = 0$. Moreover, these are evidently the only cases in which the right hand members of (6) and (7) are independent of E, G, F.

*Annali di Matematica, ser. 1, vol. 7 (1866).

§ 1. *Consideration of the form of* (1) *as restricted by the condition that* $a_1 = a_4 = 0$.

Write $$F_1(u, v) = \frac{f_1(u, v)}{f_3(u, v)}, \quad F_2(u, v) = \frac{f_2(u, v)}{f_3(u, v)}.$$

Then we find:*

$$a_1 \equiv \frac{\partial F_1}{\partial u}\frac{\partial^2 F_2}{\partial u^2} - \frac{\partial F_2}{\partial u}\frac{\partial^2 F_1}{\partial u^2}, \tag{10}$$

$$a_4 \equiv \frac{\partial F_1}{\partial v}\frac{\partial^2 F_2}{\partial v^2} - \frac{\partial F_2}{\partial v}\frac{\partial^2 F_1}{\partial v^2}. \tag{11}$$

Equating each of these to zero, and integrating, we find:

$$F_2(u, v) = \psi_1(v)\, F_1(u, v) + \psi_2(v), \tag{12}$$

$$F_2(u, v) = \phi_1(u)\, F_1(u, v) + \phi_2(u), \tag{13}$$

and from these two equations:

$$F_1(u, v) = \frac{\psi_2(v) - \phi_2(u)}{\phi_1(u) - \psi_1(v)}, \tag{14}$$

$$F_2(u, v) = \frac{\phi_1(u)\psi_2(v) - \phi_2(u)\psi_1(v)}{\phi_1(u) - \psi_1(v)}, \tag{15}$$

where $\phi_1(u)$, $\phi_2(u)$, $\psi_1(v)$, $\psi_2(v)$ are as yet arbitrary functions. Hence, when $a_1 = a_4 = 0$, equation (1) assumes the form:

$$\big(\phi_1(u) - \psi_1(v)\big) + A\,\big(\phi_1(u)\psi_2(v) - \phi_2(u)\psi_1(v)\big) + B\,\big(\psi_2(v) - \phi_2(u)\big) = 0. \tag{16}$$

§ 2. *Consideration of the values of the exponentials* $e^{\int_{a_5}^{a_2} du}$ *and* $e^{\int_{a_5}^{a_3} dv}$ *when* $a_1 = a_4 = 0$.

We have†

$$a_2 \equiv 2[F_{1u}F_{2uv} - F_{2u}F_{1uv}] + [F_{1v}F_{2uu} - F_{2v}F_{1uu}], \tag{17}$$

$$a_3 \equiv 2[F_{1v}F_{2uv} - F_{2v}F_{1uv}] + [F_{1u}F_{2vv} - F_{2u}F_{1vv}], \tag{18}$$

$$a_5 \equiv F_{1u}F_{2v} - F_{1v}F_{2u} \qquad (F_{iu} = \partial F_i/\partial u, \text{ etc.}), \tag{19}$$

Differentiating (19) with respect to u and comparing the result with (17) we find

* Previous paper, loc. cit., p. 154.

† Previous paper, loc. cit., p. 154.

$$a_2 = -a_{5u} + 3[F_{1u}F_{2uv} - F_{2u}F_{1uv}], \tag{20}$$

and in like manner, from (18) and (19), we also find

$$a_3 = a_{5v} - 3[F_{2v}F_{1uv} - F_{1v}F_{2uv}]. \tag{21}$$

From (12) and (13) we have

$$F_{2u} = \psi_1(v)F_{1u}, \tag{22}$$

$$F_{2v} = \phi_1(u)F_{1v}, \tag{23}$$

$$F_{2uv} = \psi_1(v)F_{1uv} + \psi_1'(v)F_{1u} = \phi_1(u)F_{1uv} + \phi_1'(u)F_{1v}. \tag{24}$$

These and (20) give

$$a_5 \equiv F_{1u}F_{1v}[\phi_1(u) - \psi_1(v)]. \tag{25}$$

Calculating next the bracketed expression in (21), we find it to be equal to

$$F_{1v}\{F_{1uv}[\phi_1(u) - \psi_1(v)] - F_{1u}\psi_1'(v)\} = F_{1v}\frac{\partial\left[F_{1u}\left(\phi_1(u) - \psi_1(v)\right)\right]}{\partial v}.$$

Hence we finally have

$$a_3 = a_{5v} - 3F_{1v}\frac{\partial\left[F_{1u}\left(\phi_1(u) - \psi_1(v)\right)\right]}{\partial v}. \tag{26}$$

Then from (25) and (26) we obtain:

$$\begin{aligned}\int\frac{a_3}{a_5}dv &= \int\frac{\frac{\partial a_5}{\partial v}}{a_5}dv - 3\int\frac{\frac{\partial\left[F_{1u}\left(\phi_1(u) - \psi_1(v)\right)\right]}{\partial v}}{F_{1u}\left(\phi_1(u) - \psi_1(v)\right)}dv \\ &= \log\frac{F_{1v}}{\left(F_{1u}\right)^2\left(\phi_1(u) - \psi_1(v)\right)^2}.\end{aligned} \tag{27}$$

From (14) we find:

$$\begin{aligned}F_{1v} &= \frac{[\phi_1(u) - \psi_1(v)]\psi_2'(v) + [\psi_2(v) - \phi_2(u)]\psi_1'(v)}{[\phi_1(u) - \psi_1(v)]^2}, \\ F_{1u} &= -\frac{[\phi_1(u) - \psi_1(v)]\phi_2'(u) + [\psi_2(v) - \phi_2(u)]\phi_1'(u)}{[\phi_1(u) - \psi_1(v)]^2},\end{aligned} \tag{28}$$

or say, $F_{1v} \equiv A/D^2$, and $F_{1u} \equiv B/D^2$.

Then $\int(a_3/a_5)dv$ becomes $\log(A/B^2)$. By similar reasoning we find that $\int(a_2/a_5)du = \log(-A^2/B)$.

Hence the exponentials $e^{\int\frac{a_2}{a_5}du}$ *and* $e^{\int\frac{a_3}{a_5}dv}$ *are equal, respectively, to* $-A^2/B$ *and* A/B^2, *where* A *and* B *have the meaning given in connection with* (28).

§3. *Reduction of the system of partial differential equations* (4) *under the restriction that* $a_1 = a_4 = 0$, *and conditions of integrability for the reduced system.*

Representing the right hand members of (8) and (9) by R and S respectively, and placing p for $(RS)^{\frac{1}{2}}$, we have

$$G = pE, \tag{32}$$

$$F^2 = E(p - RE^{-1}). \tag{33}$$

Also write $t = p_u E^{\frac{1}{2}} - R_u$, $h = p_v E^{\frac{1}{2}} - R_v$, $l = 2pE^{\frac{1}{2}} - \frac{3}{2}R$. Transforming system (4), we find for equation (a), after some reductions,

$$Et + (l + R - pE^{\frac{1}{2}})E_u - \sqrt{pE - RE^{\frac{1}{2}}}\,E_v = 0. \tag{34}$$

Calculation shows that *equation* (a) *reduces to* (34), and, with a little more difficulty, we find that equations (b) and (d) reduce to the same equation which is:

$$p_v RE - pER_v - pp_u E\sqrt{pE - RE^{\frac{1}{2}}}\,E_u + p(pE^{\frac{1}{2}} - \tfrac{1}{2}R)E_v = 0. \tag{35}$$

Hence system (4) *reduces, when* $a_1 = a_4 = 0$, *to equations* (34) and (35). It remains to consider the conditions of integrability for this reduced system.

We find:

$$E_u = 2\frac{E}{R^2}l, \quad E_v = 2\frac{E}{R^2}k, \tag{36}$$

where

$$l \equiv 2\Theta E^{\frac{1}{2}} - RR_u + 2\frac{\Pi}{p}\Delta, \quad k \equiv 2\Pi E^{\frac{1}{2}} - \frac{\Pi}{p}RR_u + 2\Theta\Delta,$$

in which

$$\Pi \equiv \begin{vmatrix} p & R \\ p_v & R_v \end{vmatrix}, \quad \Theta \equiv \begin{vmatrix} p & \frac{1}{2}R \\ p_u & R_u \end{vmatrix}, \quad \Delta \equiv \sqrt{pE - RE^{\frac{1}{2}}}.$$

The condition of integrability, after some reduction, can be written:

$$s^2(kl_E - lk_E) + s(l_v - k_u) + ls_v - ks_u \equiv 0. \tag{37}$$

After a somewhat long calculation we are able to write this in the form:

$$AE^{\frac{3}{2}}\Delta_E + BE\Delta_E + CE^{\frac{1}{2}}\Delta + DE^{\frac{1}{2}} + M\Delta + N + 2\frac{\Pi}{p}\Delta_v - 2\Theta\Delta_u \equiv 0, \tag{38}$$

where

$$A \equiv \frac{8}{R^2}\left(\frac{\Pi^2}{p} - \Theta^2\right), \quad B \equiv 4\frac{R_u}{R}\left(\Theta - \frac{\Pi^2}{p^2}\right), \quad C \equiv \frac{4}{R^2}\left(\Theta^2 - \frac{\Pi^2}{p}\right),$$

$$D \equiv 2\frac{R_u}{R}\left(1 - \frac{\Theta}{p}\right) + 2(\Theta_v - \Pi_u) + \frac{4}{R}(\Pi R_u - \Theta R_v),$$

$$M \equiv 2\left(\frac{\Pi}{p}\right)_v - 2\Theta_u + \Theta\frac{R_u}{R} - \frac{4}{R}\frac{\Pi}{p}R_v,$$

$$N \equiv \frac{\Pi}{p}RR_{uu} + RR_u\left(\frac{\Pi}{p}\right)_u + R_uR_v - RR_{uv} - \frac{\Pi}{p}(R_u)^2.$$

Calculating the values of Δ_E, Δ_u, Δ_v we finally have (38) in the form

$$D_1E - N_1E^{\frac{1}{2}} + (DE^{\frac{1}{2}} + N)\sqrt{pE - RE^{\frac{1}{2}}} \equiv 0, \tag{39}$$

where

$$D_1 \equiv \tfrac{1}{2}pB - \tfrac{1}{4}RA - RC + Mp + \frac{\Pi}{p}p_v - \Theta p_u,$$

$$N_1 \equiv \tfrac{1}{4}RB + RE - \frac{\Pi}{p}R_v + \Theta R_u.$$

Our conclusion is that *the necessary and sufficient conditions that the system of partial differential equations* (34) *and* (35) *shall be integrable are:*

$$D = 0, \quad D_1 = 0, \quad N = 0, \quad N_1 = 0. \tag{40}$$

§ 4. *Consideration of the form which the functions* $\phi(u)$, $\phi_1(u)$, $\phi_2(u)$, $\psi(v)$, $\psi_1(v)$, $\psi_2(v)$ *assume under the conditions of integrability.*

It is desirable to make $\phi(u)$ and $\psi(v)$, which are arbitrary, the means, as far as possible, of satisfying the conditions of integrability. If the third condition, $N = 0$, of (40) were as complicated as the other three, the problem would seem almost beyond our power. Fortunately it is somewhat simpler than the others and by means of the conclusion which we are able to reach from the condition $N = 0$, we are able so to reduce the others that they can be controlled.

Putting $a \equiv -A^2/B$ where A and B have the meaning given in connection with (28), so that

$$P \equiv \frac{a_uA_uA_v - a_uAA_{uv} - a_{uu}AA_v}{a_{uu}A^2}, \quad Q \equiv \frac{aa_{uv} - a_ua_v}{a^2a_{uu}},$$

we can, after a long reduction, put $N = 0$ in the form:

$$\frac{d\psi(v)}{dv} + 2P\psi(v) - 2Q\psi(v)^{-\frac{1}{2}} = 0, \tag{41}$$

so that

$$\psi(v)^{\frac{3}{2}} = e^{-3\int Pdv}\int e^{3\int Pdv}Qdv + C. \tag{42}$$

Calculation shows that the right hand member of (42) can be a function of N alone, only when $\partial Q/\partial u = 0$ and P is of the form $f_1(u)f_2(v)$. When Q is reduced it assumes the final form:

$$Q \equiv \frac{B(2hB^3 + kA^3)}{A^2N}, \tag{43}$$

where

$$h = \phi_1'(u)\phi_2''(u) - \phi_2'(u)\phi_1''(u), \quad k = \psi_1'(v)\psi_2''(v) - \psi_2'(v)\psi_1''(v),$$

and

$$N \equiv A^2B_{uu} - 2AB^2A_{uu} - 2B^2A_u^2 + 4ABA_uB_u - 2A^2B_u^2.$$

Since A is not a factor of B, and each is a function of both u and v (unless we consider a trivial case), it follows that A must be a factor of $2hB^3 + kA^3$. This condition is satisfied, with least restriction upon the form of the curve, by putting $h = 0$. Then as a further condition, either the product of AB and a function of v only must be equal to N, or else k must vanish, and we find that the latter includes the former. Hence Q must vanish.

The first restriction is that $\phi_1(u)'\phi_2(u)'' - \phi_2(u)'\phi_1''(u)$ *and* $\psi_1'(v)\psi_2''(v) - \psi_1'(v)\psi_2''(v)$ *each vanish.* Integrating, we have

$$\begin{aligned} \phi_2(u) &= \phi_1(u) + k_1u + k_2, \\ \psi_2(v) &= \psi_1(v) + h_1v + h_2, \end{aligned} \tag{44}$$

where h_i and k_i are to be determined. Integrating the equation $Q = 0$ with respect to v, we find, after some reduction, that the following expression must be a function of u only:

$$\frac{m_1 + m_2\psi(v)\psi'(v) + m_3v\psi_1(v) + m_4\psi_1(v) + m_5\psi_1'(v) + m_6v}{n_1 + n_2\psi(v)\psi'(v) + n_3v\psi_1(v)\psi_1'(v) + n_4v\psi(v) + h_1\phi_1'(u)v^2\psi_1''(v)},$$

where m_i and n_i do not contain v. Calculation shows that h_1 and k_1 must vanish, after which the above expression assumes the form $-2\phi_1''(u)/l\phi_1'(u)$, where $l = -k_2 + h_2$.

Calculating now the value of P, we find it to be $-2\psi_1''(v)/\psi_1'(v)$, so that

$$\frac{d\psi(v)}{dv} - 2\frac{\psi_1''(v)}{\psi_1'(v)}\psi(v) = 0,$$

or

$$\psi(v) = k[\psi_1'(v)]^2.$$

Collecting our results thus far, we have

$$\begin{aligned} \phi_2(u) &= \phi_1(u) + k_2, \\ \psi_2(v) &= \psi_1(v) + h_2, \\ \psi(v) &= k[\psi_1'(v)]^2. \end{aligned} \tag{45}$$

It remains to consider the remaining three conditions of integrability. Of these, $N_1 = 0$ is the most simple. A little consideration shows that it must

be of the form $\phi''(u) + \beta_2\phi'(u)/\beta_1 + \beta_3/\beta_1 = 0$, where β_i is independent of $\phi(u)$. It turns out that $\beta_2 \equiv 0$, and that β_3/β_1 is equal to

$$(46) \quad -\lambda k^4 \frac{[\psi_1''(v)]^2}{\psi_1(v)}\phi_1''(u) - \left\{ \frac{17}{k}\frac{[\psi_1''(v)]^2}{[\psi_1'(v)]^6} - \frac{2}{k}\frac{\psi_1'''(v)}{[\psi_1'(v)]^5} \right\}[\phi_1'(u)]^2.$$

This must be a function of u only, which requires that $\psi_1(v)$ satisfy the system:

$$(47) \quad \begin{aligned} [\psi_1''(v)]^2 &= \lambda_1\psi_1'(v), \\ 17\,[\psi_1''(v)]^2 - 2\psi_1'''(v)\psi_1'(v) &= \lambda_2[\psi_1'(v)]^6, \end{aligned}$$

where λ_i is constant. The system (47) admits of but one solution, $\lambda_i = 0$, $\psi_1''(v) = 0$. That is, $N_1 = 0$ adds to (45) the two results:

$$(48) \quad \begin{aligned} \phi(u) &= b_1u + b_2, \\ \psi_1(v) &= a_1v + a_2. \end{aligned}$$

The conditions, $D = 0$, and $D_1 = 0$, remain for consideration. The equations (45) and (48) reduce $D = 0$ to the form:

$$pR_u - p(R_u)^2 + \tfrac{1}{2}RR_u p_u = 0,$$

which is satisfied by either $R_u = 0$, or $p - pR_u + \frac{1}{2}Rp_u = 0$. A little calculation shows that $D_1 = 0$ is also satisfied if $R_u = 0$. It remains to determine whether $D_1 = 0$ is also satisfied when $\Theta = p$. We find that $D_1 = 0$ assumes the form:

$$pR_u - Rp_u = \tfrac{2}{3}p,$$

which we are to consider simultaneously with $pR_u - \frac{1}{2}Rp_u = p$. This requires either that $p = 0$, or that $R_u = \frac{4}{3}$.

If $p = 0$, then, either $G = 0$, or $EG - F^2 = 0$, the latter of which is excluded. If $G = 0$ we find that $E = 1/R^2$ and $F = \pm 1$, and the corresponding surface is an imaginary ruled surface.

The assumption $R_u = 0$ would lead us to the relation

$$(b_1u + b_2)^2(\tfrac{4}{3}u + \delta)^3 = \lambda^4k^2a_1^{12}c^2,$$

which would require that $\phi(u)$ vanish, and hence lead us to the same imaginary surface as for $p = 0$.

Hence: *In order that a real surface exist, it is necessary and sufficient that we have:*

$$(49) \quad \begin{aligned} \phi\ (u) &= b_1u + b_2, & \psi\ (v) &= ka_1^2, \\ \psi_1(v) &= a_1v + a_2, & \phi_1\,(u) &= c_1u + c_2, \\ \psi_2(v) &= \psi_1(v) + h_2, & \phi_2\,(u) &= \phi_1(u) + k_2, \end{aligned}$$

where the constants must not cause $\phi(u)$ or $\psi(v)$ to vanish.

If now we calculate a_2 and a_3 we find that they also vanish, so that we are led to the conclusion that *there is for our proposed problem no new solution.*

§ 5. *Integration of the system of partial differential equations.*

The conditions of integrability (40) being satisfied, we proceed to integrate the corresponding systems, (34) and (35), of partial equations.

They may now be written:

$$E_u = -2nE^{\frac{1}{2}}, \quad E_v = -2nE^{\frac{1}{2}}\sqrt{p - RE^{-1}}, \tag{50}$$

where n is the constant $b_1/(k_2 - h_2)k^{\frac{1}{2}}a_1c_1$.

The integrals of these are, respectively,

$$E^{-\frac{1}{2}} = nu + f_1(v), \quad \sqrt{p - RE^{-1}} = -\tfrac{1}{2}nv + f_2(u), \tag{51}$$

where $f_1(v)$ and $f_2(u)$ are yet to be determined.

We have the identical relation:

$$p - [f_2(u) - \tfrac{1}{2}nv]^2 \equiv R[nu + f_1(v)].$$

Writing $u = 0$, we have at once:

$$f_1(v) \equiv \frac{p_0 - [f_2(0) - \frac{1}{2}nv]^2}{R},$$

and further consideration shows that $f_2(u) \equiv f_2(0)$, a constant which we denote by δ. Writing $D \equiv m(b_1u + b_2) - [\delta - \frac{1}{2}nv]^2$, we finally have:

$$E = \frac{R^2}{D^2}, \quad F = \frac{R(\delta - \frac{1}{2}nv)}{D}, \quad G = \frac{m(b_1u + b_2)R^2}{D^2}. \tag{52}$$

§ 6. *Curvature of the surface corresponding to the problem considered.*

The Gaussian curvature of any surface is given by:

$$K = \frac{1}{\sqrt{EG - F^2}}\left[\frac{\partial}{\partial v}\left(\frac{\sqrt{EG - F^2}}{E}\begin{Bmatrix}11\\2\end{Bmatrix}\right) - \frac{\partial}{\partial u}\left(\frac{\sqrt{EG - F^2}}{G}\begin{Bmatrix}12\\2\end{Bmatrix}\right)\right],$$

where,

$$\begin{Bmatrix}11\\2\end{Bmatrix} \equiv \frac{-FE_u + 2EF_u - EE_v}{2(EG - F^2)}, \quad \begin{Bmatrix}12\\2\end{Bmatrix} = \frac{EG_u - FE_v}{2(EG - F^2)}.$$

Referring to system (4) we find that $\begin{Bmatrix}11\\2\end{Bmatrix} = \frac{a_4}{a_5}$. Writing $EG - F^2 = \Delta$, we have, from the same system,

$$\frac{EG_u - FE_v}{\Delta} = \frac{a_2}{a_5} + \frac{1}{2}\frac{GE_u}{\Delta} + \frac{FE_v - FF_u}{\Delta},$$

$$-\frac{F}{E}\frac{a_1}{a_5}=\tfrac{1}{2}\frac{FE_v-FF_u}{\Delta}+\tfrac{1}{2}\frac{F^2}{E}\frac{E_u}{\Delta},$$

and from these,

$$\begin{Bmatrix}12\\2\end{Bmatrix}=\tfrac{1}{2}\frac{a_2}{a_5}-\tfrac{1}{2}\frac{F}{E}\frac{a_1}{a_5}+\tfrac{1}{4}\frac{E_u}{E}.$$

Put

$$M=\frac{G^{\frac{1}{2}}}{\phi(u)^{\frac{1}{2}}E},\quad N=\frac{\psi^{\frac{1}{2}}}{E^{\frac{1}{2}}},$$

$$\alpha=-\tfrac{1}{2}\frac{a_3}{a_5}+\tfrac{3}{2}\frac{a_4}{a_5}\frac{F}{G},\quad \beta=\tfrac{1}{2}\frac{a_2}{a_5}-\tfrac{3}{2}\frac{a_1}{a_5}\frac{F}{E},$$

$$H=\frac{\sqrt{\Delta}}{E},\quad \begin{Bmatrix}11\\2\end{Bmatrix}=W,\quad \begin{Bmatrix}12\\2\end{Bmatrix}=U.$$

Then from (6) and (7), we have

$$H=Me^{\int\alpha dv}=Ne^{\int\beta du},$$

and some calculation gives:

$$K=\frac{1}{E}\left\{\frac{\partial}{\partial v}\left(\frac{a_1}{a_5}\right)-U_u+\begin{vmatrix}\frac{a_1}{a_5} & \beta+\frac{\partial\log N}{\partial u}\\ U & \alpha+\frac{\partial\log M}{\partial N}\end{vmatrix}\right\}.$$

Then calculating U_u, $\partial\log N/\partial u$, and $\partial\log M/\partial v$ we find as a final expression for *the total curvature of a surface corresponding to the system of partial differential equations* (4)*:*

$$K=\frac{1}{E}\left\{\frac{\partial}{\partial v}\left(\frac{a_1}{a_5}\right)-\tfrac{1}{2}\frac{\partial}{\partial u}\left(\frac{a_2}{a_5}\right)-\tfrac{1}{4}\frac{\partial^2\log E}{\partial u_2}+\frac{F}{E}\frac{\partial}{\partial u}\left(\frac{a_1}{a_5}\right)+\tfrac{1}{2}\frac{a_1}{a_5}\frac{EF_u-FE_u}{E^2}\right\}$$

(53)

$$+\frac{1}{E}\begin{vmatrix}\frac{a_1}{a_5} & \frac{1}{2}\frac{a_2}{a_5}-\frac{3}{2}\frac{a_1}{a_5}\frac{F}{E}-\frac{1}{4}\frac{E_u}{E}\\ \frac{1}{2}\frac{a_2}{a_5}-\frac{1}{2}\frac{a_1}{a_5}\frac{F}{E}+\frac{1}{4}\frac{E_u}{E} & -\frac{1}{2}\frac{a_3}{a_5}+\frac{3}{2}\frac{a_4}{a_5}\frac{F}{G}+\frac{3}{4}\frac{G_v}{G}-\frac{E_v}{E}\end{vmatrix}.$$

If $a_1=a_4=0$, this reduces to:

$$K=\frac{1}{E}\left\{-\tfrac{1}{2}\frac{\partial}{\partial u}\left(\frac{a_2}{a_5}\right)-\tfrac{1}{4}\frac{\partial^2\log E}{\partial u^2}-\tfrac{1}{4}\left(\frac{a_2}{a_5}\right)^2+\tfrac{1}{16}\left(\frac{E_u}{E}\right)^2\right\}.$$

But then, $E=R^2/(A-B^2)^2$, where $A=mb_1u+mb^2$ and $B=\delta-\frac{1}{2}nv$.

Hence:

$$\tfrac{1}{16}\left(\frac{E_u}{E}\right)^2 - \tfrac{1}{4}\frac{\partial^2 \log E}{\partial u^2} = -\tfrac{1}{4}\frac{m^2 b_1^2}{(A - B^2)^2}.$$

Making use of this relation we have:

$$K = \frac{1}{E}\left[-\tfrac{1}{2}\frac{\partial}{\partial u}\left(\frac{a_2}{a_5}\right) - \tfrac{1}{4}\left(\frac{a_2}{a_5}\right)^2\right] - \tfrac{1}{4}\frac{m^2 b_1^2}{R^2}.$$

Here we notice that if a_2 were zero the curvature would be constant—which agrees with known results.

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ZUR ERKLÄRUNG DER BOGENLÄNGE UND DES INHALTES EINER KRUMMEN FLÄCHE*

VON

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1. Dass die Zahl, welche nach Festsetzung der Längeneinheit einer begrenzten krummen Linie zuzuordnen ist, der Erklärung bedürfe, ist allgemein anerkannt. Es ist übrigens die Aufstellung einer solchen Erklärung ziemlich naheliegend. Denken wir uns z. B. für jeden Werth von x im endlichen Intervalle (a, b) eine reelle Function $y = f(x)$ eindeutig definirt und $x\,y$ als rechtwinkliche Coordinaten in der Ebene construirt, so wird die Länge des von den Punkten

$$A \equiv \big(a, f(a)\big), \qquad B \equiv \big(b, f(b)\big),$$

begrenzten Bogens folgendermassen erklärt. Wir theilen $b - a$ in n Theile $\delta_1, \delta_2, \cdots, \delta_n$, sodass

$$\delta_1 + \delta_2 + \cdots + \delta_n = b - a$$

ist, und setzen $a = a_0$, $a_0 + \delta_1 = a_1$ u. s. f., allgemein $a_{r-1} + \delta_r = a_r$ und schliesslich $b = a_n = a_{n-1} + \delta_n$. Die den Absscissen $a_1 \cdots a_{n-1}$ entsprechenden Punkte des Bogens AB seien mit $A_1 \cdots A_{n-1}$ bezeichnet. Wenn alsdann

$$\sum_{r=1}^{n} |A_{r-1}A_r| = \sum_{r=1}^{n} \sqrt{\delta_r^2 + \{f(a_r) - f(a_{r-1})\}^2}$$

bei unbeschränkter und unbegrenzter Abnahme aller Theile $\delta_1 \cdots \delta_n$, deren Summe jedoch stets $b - a$ bleiben muss, einen endlichen Grenzwerth L besitzt d. h. *wenn es eine Zahl L von der Beschaffenheit giebt, dass jedem $\epsilon > 0$ eine positive Zahl Δ in der Art entspricht, dass wenn nur jede der Zahlen $\delta_1 \cdots \delta_n$, deren Summe $b - a$ ist, kleiner als Δ ist, dann stets*

$$(1) \qquad \Big| L - \sum_{r=1}^{n} \sqrt{\delta_r^2 + \{f(a_r) - f(a_{r-1})\}^2} \Big| < \epsilon,$$

ist †—so ordnen wir dem Bogen AB diese Zahl L zu.

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† Der Kürze wegen bezeichnen wir diese Erklärung als "Erklärung (D)".—Aus ihr dürfte noch nicht hervorgehen, dass wenn C einen Punkt des Bogens AB zwischen A und B bezeichnet der Bogen AC ebenfalls rectificirbar sei.

Lassen wir die Function $f(x)$ in jedem Punkte x des Intervalles (a, b) mit Einschluss von $x = a$ und $x = b$ stetig sein und versehen sie in allen diesen Punkten mit einem vollständigen Differentialquotienten $f'(x)$, der ebenfalls bei jedem von ihnen stetig sein soll, so erfüllt das eigentliche bestimmte Integral

$$\int_a^b \sqrt{1 + f'(x)^2}\, dx, \tag{2}$$

die oben an die Zahl L gestellte Bedingung, wie aus der Erklärung desselben unmittelbar hervorgeht. Das Nämliche gilt, wenn der wenigstens für alle x im Intervalle (a, b) mit Ausnahme eines Werthsystems 1. Gattung vorhandene vollständige Differentialquotient $f'(x)$ eine im Intervalle (a, b) endliche und integrirbare Function bildet.

2. Schwieriger gestaltet sich der Nachweis dieses Satzes schon in dem Falle, dass der Differentialquotient $f'(x)$ für alle Werthe x im Intervalle (a, b) mit Ausnahme eines der Endwerthe, etwa von $x = a$, stetig, dabei aber *in diesem Intervalle nicht endlich ist, während das Integral* (2) *einen Sinn hat.* Zunächst wird man folgendes bemerken.* Vermöge der Stetigkeit der Function $f(x)$ bei $x = a$ lässt sich der beliebig vorgegebenen Zahl $\epsilon/2$ eine positive Zahl δ so zuordnen, dass

$$|AM| = \sqrt{(x-a)^2 + \{f(x) - f(a)\}^2} < \frac{\epsilon}{2} \tag{3}$$

ist, wenn nur $0 < x - a < \delta$ ist. Wegen der Existenz des Integrals (2) dürfen wir ferner annehmen, dass unter der nämlichen Voraussetzung

$$0 < \int_a^x \sqrt{1 + f'(x)^2}\, dx < \frac{\epsilon}{2} \tag{4}$$

ist. Wir setzen nun $\delta_1 = \delta$. Dann ist zufolge der Ungleichungen (3) und (4)

$$-\frac{\epsilon}{2} < \int_a^{a_1} \sqrt{1 + f'(x)^2}\, dx - |AA_1| < \frac{\epsilon}{2} \qquad (a_1 = a + \delta). \tag{5}$$

Ist δ in dieser Weise festgelegt, so gehört nach dem oben Bemerkten zu $\epsilon/2$ eine solche positive Zahl κ, dass wenn nur die Theile $\delta_2, \delta_3, \cdots, \delta_n$ mit der Summe $(b - a) - \delta$ sämmtlich kleiner als κ sind, alsdann

$$0 < \int_{a_1}^b \sqrt{1 + f'(x)^2}\, dx - \sum_{r=2}^{n} |A_{r-1}A_r| < \frac{\epsilon}{2} \tag{6}$$

* Von mir bereits hervorgehoben, Mathematische Annalen, Bd. 18, S. 274, jedoch in die *Grundzüge der Differential- und Integralrechnung* (II, S. 310) nicht aufgenommen.

ist. Durch Addition der Beziehungen (5) und (6) findet man, dass unter den angegebenen Voraussetzungen über δ_1 einer- und $\delta_2 \cdots \delta_n$ andererseits

$$\left| \int_a^b \sqrt{1 + f'(x)^2}\, dx - \sum_{r=1}^{n} A_{r-1}A_r \right| < \epsilon \tag{7}$$

ist. Allein dieses Ergebniss deckt sich nicht völlig mit der Erklärung (D), indem hier nicht für alle Theile $\delta_1 \cdots \delta_n$ *eine* obere Grenze Δ erhalten wird, sondern erst nach Festsetzung von δ_1 für die übrigen Theile $\delta_2 \cdots \delta_n$ eine *von* δ_1 *abhängige* obere Grenze κ.

Will man die Gewissheit erlangen, dass das Integral (2) auch jetzt die in der Erklärung (D) an die Zahl L gerichtete Forderung befriedigt, so muss man noch nachweisen, dass für die Summe $\sum_r |A_{r-1}A_r|$ bei $\lim \delta_r = 0$ $(r = 1, \cdots, n)$ überhaupt ein endlicher Grenzwerth vorhanden ist.* Man hat demnach zu zeigen, dass jedem $\epsilon > 0$ ein $\Delta > 0$ so entspricht, dass wenn nur alle δ_r und alle δ'_r $(r = 1, 2, \cdots, n')$ kleiner als Δ sind, alsdann stets

$$\left| \sum_{r=1}^{n} |A_{r-1}A_r| - \sum_{r=1}^{n'} |A'_{r-1}A'_r| \right| < \epsilon$$

ist. Dazu könnte, im Falle dass der Bogen AB convex ist, meine Entwickelung in den Mathematischen Annalen, Bd. 28, S. 271 dienen. Dieser Grenzwerth muss zufolge der Beziehung (7) das Integral (2) sein.

3. Die in der vorigen Nr. angedeutete Entwickelung lässt sich dadurch umgehen, dass man für die dort betrachtete Curve eine andere analytische Darstellung verwendet. Man wird es leicht so einrichten können, dass x und y als eindeutige Functionen eines Parameters t

$$x = \phi(t), \quad y = \psi(t) \qquad (\alpha \leqq t \leqq \beta), \tag{a}$$

erscheinen und zwar als für alle Werthe von t im endlichen Intervalle (α, β) stetige Functionen, welche, etwa abgesehen von einer endlichen Anzahl von diesen Werthen, für jeden von ihnen vollständige und zwar stetige Differentialquotienten $\phi'(t)$ $\psi'(t)$ besitzen, welche im genannten Intervalle endlich sind.

* Da die Summen $\sum_r |A_{r-1}A_r|$ endlich sind (nämlich unter der Zahl

$$M + \int_a^b \sqrt{1 + f'(x)^2}\, dx,$$

worin M den grössten Abstand der Punkte des Bogens AB von seinem Anfangspunkte A bedeutet, liegen), so könnten sie bei $\lim \delta_r = 0$ $(r = 1, \cdots, n)$ auch zwei positive, von einander verschiedene Unbestimmtheitsgrenzen haben.

Die Erklärung der Länge des durch die Gleichungen (a) dargestellten Bogens AB ist die unmittelbare Verallgemeinerung der Erklärung (D). Indem wir jetzt

$$\beta - a = \delta_1 + \delta_2 + \cdots + \delta_n,$$

$$a + \delta_1 = a_1, \quad a_1 + \delta_2 = a_2, \cdots, \quad a_{n-1} + \delta_n = a_n = \beta$$

und

$$\sqrt{[\phi(a_r) - \phi(a_{r-1})]^2 + [\psi(a_r) - \psi(a_{r-1})]^2} = |A_{r-1}A_r|$$

setzen, so heisst L die Zahl des genannten Bogens, falls jedem $\epsilon > 0$ ein $\Delta > 0$ so zugeordnet werden kann, dass stets

$$(b) \qquad \left| L - \sum_{r=1}^{n} |A_{r-1}A_r| \right| < \epsilon$$

ausfällt, wenn nur eine jede Strecke $\delta_1 \cdots \delta_n$ kleiner als Δ ist.

Wir haben gezeigt,* dass wenn $\phi'(t)\ \psi'(t)$ bei jedem Werthe des Intervalls (a, β), $t = a$ und $t = \beta$ also eingeschlossen, stetig ist, alsdann das Integral

$$(c) \qquad \int_a^\beta \sqrt{\phi'(\tau)^2 + \psi'(\tau)^2}\, d\tau \quad (= J)$$

die soeben hinsichtlich der Zahl L aufgestellte Bedingung erfüllt. Dieser Beweis lässt sich jedoch nicht auf den Fall ausdehnen, dass $\phi'(t)$ und $\psi'(t)$ unbeschadet ihrer Endlichkeit im Intervalle (a, β) Unstetigkeiten für einzelne Werthe desselben erleiden.†

Es giebt indess ein sehr bequemes Verfahren, den Satz, dass das Integral (c) die Länge des Curvenbogens (a) sei, in den beiden soeben erwähnten Fällen zu begründen.‡ Wir wollen dasselbe in Folgendem etwas eingehender auseinandersetzen.

4. *Satz.* „*Es sei* $f(x_1 x_2 \cdots x_k)$ *eine für jedes System von Werthen* $x_1 x_2 \cdots x_k$, *wovon*

$$(1) \qquad a \leqq x_1 \leqq \beta, \quad a \leqq x_2 \leqq \beta, \cdots, a \leqq x_k \leqq \beta$$

ist, eindeutige und für diesen Bereich endliche Function von $x_1 \cdots x_k$. Wir schalten zwischen a und β die $n - 1$ Werthe $a_1 \cdots a_{n-1}$ ein und setzen

$$a_r - a_{r-1} = \delta_r \qquad (r = 1, 2, \cdots, n),$$

wobei $a_0 = a$, $a_n = \beta$ sein soll. Versteht man dann unter f_r irgend einen Werth zwischen der oberen und unteren Grenze von $f(x_1 x_2 \cdots x_k)$ unter der Vor-

* Vgl. *Grundzüge* u. s. w., II, S. 314.

† A. a. O. ist die Giltigkeit dieses Beweises für den zuletzt erwähnten Fall in Folge eines Rechenfehlers fälschlich behauptet.

‡ Angedeutet ist es Mathematische Annalen, Bd. 18, S. 274; *Grundzüge*, II, S. 316, Note.

aussetzung dass $x_1 x_2 \cdots x_k$ je einen beliebigen Werth im Intervalle (a_{r-1}, a_r) bedeuten (diese Grenzen selbst eingeschlossen) und hat $\sum_r f_r \delta_r$ bei $\lim \delta_r = 0$ $(r = 1, 2, \cdots, n)$ *einen endlichen Grenzwerth* J, *so ist die Function* $f(t\,t \cdots t)$ *im Intervalle* (α, β) *integrirbar und zwar ist*

$$\int_\alpha^\beta f(t\,t \cdots t)\,dt = J."$$

Der Satz ist selbstverständlich; denn man darf unter den angegebenen Umständen auch

$$(1^*) \qquad \mathop{\mathrm{L}}_{\delta_r = 0} \sum_{r=1}^{n} f(x_r x_r \cdots x_r)\,\delta_r = J$$

setzen, unter x_r einen beliebig im Intervalle (a_{r-1}, a_r) zu wählenden Werth verstanden.

5. Es erhebt sich nunmehr die Frage, unter welchen Umständen die genannte Summe $\sum_r f_r \delta_r$ bei $\lim \delta_r = 0$ einen endlichen Grenzwerth J besitzt d. h. es eine Zahl J von der Beschaffenheit gibt, dass jedem $\epsilon > 0$ ein $\Delta > 0$ so entspricht, dass wenn nur jeder der Theile $\delta_1 \cdots \delta_n$, deren Summe $\beta - \alpha$ ist, kleiner als Δ ist, dann stets

$$\left| J - \sum_{r=1}^{n} f_r \delta_r \right| < \epsilon$$

ist.—Bezeichnen wir die obere und untere Grenze von $f(x_1 x_2 \cdots x_k)$ im Bereiche

$$(2) \qquad a_{r-1} \leqq x_1 \leqq a_r, \quad a_{r-1} \leqq x_2 \leqq a_r, \cdots, \quad a_{r-1} \leqq x_k \leqq a_r$$

bezw. mit g_r k_r, so soll $g_r \geqq f_r \geqq r_r$ sein, so dass

$$(3) \qquad \sum_{r=1}^{n} g_r \delta_r \geqq \sum_{r=1}^{n} f_r \delta_r \geqq \sum_{r=1}^{n} k_r \delta_r$$

ist.

Die Summe $\sum_r g_k \delta_r$ *hat nun bei* $\lim \delta_r = 0$ $(r = 1, \cdots, n)$ *einen endlichen Grenzwerth* G. Dies ergiebt sich genau auf die nämliche Weise, wie die Existenz des oberen Integrals einer im Intervalle (α, β) von x eindeutigen und endlichen Function von x.* In der That bestehen die drei folgenden Sätze:

1) Betrachten wir zunächst nun *ein* unendliches System solcher Theilungen $\mathfrak{T}_1 \mathfrak{T}_2 \cdots \mathfrak{T}_m \cdots$ des Unterschiedes $\beta - \alpha$, dass kein Theil von $\mathfrak{T}_m$ in zwei benachbarten Theilen von $\mathfrak{T}_{m-1}$ liegt, und dabei, wenn nur m gross genug genommen wird, jeder zu $\mathfrak{T}_m$ gehörige Theil kleiner als eine vorgegebene Zahl Δ ausfällt, und bezeichnen die zur Theilung $\mathfrak{T}_m$ gehörige Summe $\sum_r g_r \delta_r$ mit Σ_m, so nimmt Σ_m mit wachsendem m nicht zu und hat daher bei $\lim m = +\infty$ einen endlichen Grenzwerth G.

* Vgl. des Verfassers *Grundzüge* I, S. 353; III, S. 264.

2) Wird $\beta - a$ auf irgend eine Weise in n Theile $\delta_1 \cdots \delta_n$ zerlegt, *so ist* $\sum_r g_r \delta_r$ *nicht kleiner als die in* 1) *eingeführte Zahl* G.

3) Diese Zahl G ist der Grenzwerth von $\sum_r g_r \delta_r$ bei $\lim \delta_r = 0$ $(r = 1, \cdots, n)$ d. h. jedem $\epsilon > 0$ lässt sich ein $\Delta > 0$ so zuordnen, dass, wenn nur ein jeder der Theile δ_r, deren Summe $\beta - a$ ist, kleiner als Δ ist, dann stets

$$0 \leqq \sum_{r=1}^{n} g_r \delta_r - G < \epsilon$$

ist.

Auf die gleiche Weise ergiebt sich, dass die Summen $\sum_r k_r \delta_r$ bei $\lim \delta_r = 0$ $(r = 1, \cdots, n)$ einen endlichen Grenzwerth K hat, welcher zu Folge der Beziehungen (3) nicht grösser als G sein kann.

Dazu dass die Summe $\sum_r f_r \delta_r$ bei willkürlicher Annahme der Zahlen f_r innerhalb der Grenzen k_r und g_r für die unabhängig von einander zur Null convergirenden Strecken $\delta_1 \cdots \delta_n$ *einen und derselben endlichen Grenzwerth* J *besitze, ist nothwendig und hinreichend,* dass $G = K$ oder

$$(3^*) \qquad \mathop{\mathrm{L}}_{\delta_r = 0} \sum_{r=1}^{n} (g_r - k_r) \delta_r = 0$$

ist. Alsdann ist $J = G = K$. Dies folgt aus den Beziehungen (3) unmittelbar.

Wenn die Function $f(x_1 \cdots x_k)$ *für jedes System* $x_1 x_2 \cdots x_k$ *im Bereiche* (1) *stetig ist,* so existirt für $\sum_r f_r \delta_r$ bei $\lim \delta_r = 0$ $(r = 1, \cdots, n)$ *ein* Grenzwerth J. Unter dieser Voraussetzung besteht nämlich die Gleichung (3*), wie sich aus der sogenannten gleichmässigen Stetigkeit der Function $f(x_1 \cdots x_k)$ im Bereiche (1) d. i. aus dem Satze; „Jedem $\epsilon > 0$ entspricht ein $\Delta > 0$ so dass

$$|f(x_1' \cdots x_k') - f(x_1 \cdots x_k)| < \epsilon$$

ist, wenn nur $x_1 \cdots x_k$ und $x_1' \cdots x_k'$ zwei Stellen des Bereiches (1) bedeuten, wofür jede der Differenzen $x_1' - x_1 \cdots x_k' - x_k$ dem Betrage nach kleiner als Δ ist" in bekannter Weise * ergiebt.

6. Mit Hilfe des Satzes in Nr. 4 lässt sich der Grenzwerth der in Nr. 3 eingeführten Summe

$$(4) \qquad \sum_{r=1}^{n} |A_{r-1} A_r|,$$

bei $\lim \delta_r = 0$ im Falle, dass *nicht allein die Functionen* $\phi(t)$, $\psi(t)$ *sondern auch ihre Differentialquotienten* $\phi'(t)$ $\psi'(t)$ *im Intervalle* (a, β) *von* t *überall stetig sind,* unmittelbar bestimmen. Nunmehr ist

$$(5) \qquad \phi(a_r) - \phi(a_{r-1}) = \phi'(t_r)\,\delta_r, \quad \psi(a_r) - \psi(a_{r-1}) = \psi'(u_r)\,\delta_r,$$

* Vgl. z. B. *Grundzüge* I, S. 357.

wobei t_r u_r nicht näher bekannte Werthe innerhalb des Intervalles (a_{r-1}, a_r) bezeichnen. Demnach ist

$$|A_{r-1}A_r| = \sqrt{\phi'(t_r)^2 + \psi'(u_r)^2}\,\delta_r \qquad (r = 1, 2, \cdots, n). \tag{6}$$

Nehmen wir jetzt die Function

$$f(x_1, x_2) = \sqrt{\phi'(x_1)^2 + \psi'(x_2)^2}$$

zu Hilfe, welche für jedes Werthsystem $x_1 x_2$ im Bereiche

$$\alpha \leqq x_1 \leqq \beta, \qquad \alpha \leqq x_2 \leqq \beta,$$

stetig ist, so hat nach Nr. 5 die Summe $\sum_r f_r \delta_r$ bei $\lim \delta_r = 0$ einen endlichen Grenzwerth. Er ist nach Nr. 4 das Integral (*c*) in Nr. 3. Zufolge der Gleichung (6) ist die Summe (4) ein besonderer Fall der soeben aufgestellten Summe $\sum_r f_r \delta_r$; wir finden somit

$$\mathop{\mathrm{L}}_{\delta_r = 0} \sum_{r=1}^{n} |A_{r-1}A_r| = \int_\alpha^\beta \sqrt{\phi'(t)^2 + \psi'(t)^2}\,dt. \tag{7}$$

Das Integral rechts ist demnach in dem hier besprochenen Falle die Länge des durch die Gleichungen (*a*) dargestellten ebenen Bogens AB.

7. Es ist unerlässlich die Formel (7) mindestens auf den Fall auszudehnen, dass *das Intervall* (α, β) *bloss in eine endliche Anzahl von Theilen zerfällt, in denen jedem sowohl die Function* $\phi'(t)$, *als auch* $\psi'(t)$ *ausnahmslos stetig ist,* während für die Grenzpunkte dieser Theile mindestens eine der Functionen $\phi(t)$, $\psi(t)$ keinen vollständigen Differentialquotienten hat.

Denken wir uns zwischen α und β nacheinander die festen Werthe $\gamma_1 \gamma_2 \cdots \gamma_m$ eingeschaltet und nehmen an, dass in den $m + 1$ Intervallen $(\alpha, \gamma_1), (\gamma_1, \gamma_2), \cdots, (\gamma_m, \beta)$ von t die beiden Functionen $\phi'(t)$, $\psi'(t)$ allenthalben stetig seien. Es sollen also $\phi'(t)$ und $\psi'(t)$ sowohl bei $\lim t = \gamma_s - 0$, als auch bei $\lim t = \gamma_s + 0$ endliche Grenzwerthe haben, welche aber von einander verschieden sind. Kommen unter den zwischen α und β eingeschalteten Werthen $a_1 \cdots a_n$ stets $\gamma_1 \gamma_2 \cdots \gamma_m$ vor, so zerfällt die Summe (4) einfach in $(m + 1)$ Glieder, welche den soeben erwähnten Theilen des Intervalles (α, β) entsprechen. Auf jeden von ihnen findet die Formel (7) sinngemässe Anwendung, sodass sich unter dieser besonderen Voraussetzung über die Theilstrecken $\delta_1 \cdots \delta_n$ die Formel ergiebt,

$$\mathop{\mathrm{L}}_{\delta_r = 0} \sum_{r=1}^{n} |A_{r-1}A_r| = \sum_{r=0}^{m} \int_{\gamma_r}^{\gamma_{r+1}} \sqrt{\phi'(t)^2 + \psi'(t)^2}\,dt = \int_\alpha^\beta \sqrt{\phi'(t)^2 + \psi'(t)^2}\,dt \tag{8}$$

$$(\gamma_0 \equiv \alpha,\ \gamma_{m+1} \equiv \beta).$$

Nun wollen wir auch solche Theilungen $\delta_1 \cdots \delta_n$ von $\beta - \alpha$ berücksichtigen, dass einige oder alle Werthe $\gamma_1 \cdots \gamma_m$ in das Innere von gewissen unter den δ_r

fallen. Ist δ_s eine derartiger Theil, so kann eine oder auch beide Formeln (5) für $r = s$ ihre Giltigkeit verlieren. Man darf jedoch auch jetzt noch

$$\phi(a_s) - \phi(a_{s-1}) = \phi'_s \delta_r, \quad \psi(a_s) - \psi(a_{s-1}) = \psi'_s \delta_r,$$

wo ϕ'_s ψ'_s Mittelwerthe zwischen den oberen und unteren Grenze von $\phi'(t)$ bezw. $\psi'(t)$ im Intervalle (a_{s-1}, a_s) von t bedeuten, und daher

$$|A_{s-1}A_s| = \sqrt{\phi'^2_s + \psi'^2_s}\,\delta_s \tag{9}$$

setzen. Doch ist es nicht sicher, ob

$$\sqrt{\phi'^2_s + \psi'^2_s}$$

zwischen der oberen und unteren Grenze der obigen Function $f(x_1, x_2)$ im Bereiche

$$a_{s-1} \leqq x_1 \leqq a_s, \quad a_s \leqq x_2 \leqq a_s$$

liegt. Der Satz in Nr. 4 reicht also für die hier zu Grunde gelegte Annahme über die Functionen $\phi(t)$ $\psi(t)$ nicht aus.

8. Man kann indess den Satz in Nr. 4 in der Weise verallgemeinern, dass man an Stelle der dort angebenen Summe

$$\sum_{r=1}^{n} f'_r \delta_r$$

setzt, wobei für alle Werthe von r, denen ein Theil δ_r entspricht, der in seinem Innern einen der Punkte $\gamma_1 \cdots \gamma_m$ nicht enthält, $f'_r = f_r$ zu denken ist, während einem jeden Theil δ_s, zu dessen Innerm ein solcher Werth gehört, eine willkürliche Zahl f'_s zwischen zwei nach Belieben festgesetzten Grenzen zugeordnet werden soll.—Die Gleichung (1*) besteht nämlich noch immer, indem ja x_s von dem im Intervalle (a_{s-1}, a_s) vorkommenden der Werthe $\gamma_1 \cdots \gamma_m$ verschieden gedacht werden kann.

Wir haben also

$$\mathbf{L}_{\delta_r = 0} \sum_{r=1}^{n} f'_r \delta_r = J. \tag{10}$$

Die Summe $\sum f'_r \delta_r$ hat aber bei $\lim \delta_r = 0$ sicher einen endlichen Grenzwerth, wenn das nämliche von der in Nr. 4 eingeführten Summe $\sum f_r \delta_r$ gilt, und zwar sind diese beiden Grenzwerthe einander gleich. Wir nehmen also an, dass es eine Zahl J von der Beschaffenheit gebe, dass zu jedem $\epsilon > 0$ ein $\Delta > 0$ so gehört, dass neben

$$\delta_r < \Delta \ (r = 1 \cdots n) \quad \text{stets} \quad \left| \sum_{r=1}^{n} f_r \delta_r - J \right| < \epsilon$$

ist. Nun ist

$$\sum_{r=1}^{n} f'_r\delta_r - J = \left\{ \sum_{r=1}^{n} f_r\delta_r - J \right\} + \sum_{s} (f'_s - f_s)\,\delta_s, \tag{11}$$

wobei sich die Summe nach s auf die Zeiger derjenigen δ_r bezieht, in denen einer der Werthe $\gamma_1 \cdots \gamma_m$ vorkommt. Jeder zu einer Stelle des Bereiches (1) gehörige Werth von $f(x_1 \cdots x_k)$ liegt zwischen dieselben Zahlen AB, so dass auch $A < f_s < B$ ist. Für die m Werthe f'_s seien die Grenzen A' und B' festgesetzt, es sei also $A' \leqq f'_s \leqq B'$.

Somit ist

$$A' - B < f'_s - f_s < B' - A \qquad \text{d. h.} \qquad |f_s - f'_s| < C,$$

unter C die grössern der Zahlen $|A' - B|$, $|B' - A|$ verstanden. Wir finden also, wenn wir mit δ die grösste unter den Strecken δ_r bezeichnen,

$$\left| \sum_{s} (f'_s - f_s)\,\delta_s \right| < mC\delta.$$

Lassen wir nun alle Theile δ_r sowohl kleiner als Δ, als auch kleiner als ϵ/mC sein, so ist nach (11) stets

$$\left| \sum_{r=1}^{n} f'_r\delta_r - J \right| < 2\epsilon.$$

d. h. es besteht die Gleichung (10).

9. Wir wollen nun zeigen, dass *die Summe* $\sum_r f_r d_r$ *auch dann bei* lim $\delta_r = 0$ $(r = 1, \cdots, n)$ *einen endlichen Grenzwerth besitzt, wenn die Function* $f(x_1 \cdots x_k)$ *unbeschadet ihrer Endlichkeit im Bereiche* (1) *bloss bei denjenigen Stellen* $x_1 \cdots x_k$, *zu welchen keiner der Werthe* $\gamma_1 \cdots \gamma_m$ *gehört, stetig ist.*

Zu diesem Behufe haben wir aus den Ergebnissen von No. 5 noch den Satz herzuleiten: „Die Gleichung (3*) gilt sicher, wenn wir jeder vorgegebenen Zahl $\epsilon > 0$ *mindestens eine* Theilung $\delta_1 \cdots \delta_n$ des Unterschiedes $\beta - a$ zuzuweisen vermögen, wofür

$$0 \leqq \sum_{r=1}^{n} (g_r - k_r)\,\delta_r < \epsilon \tag{12}$$

ist." Da wir von vorneherein wissen, dass die in (12) stehende Summe bei lim $\delta_r = 0$ den Grenzwerth $G - K$ hat, so müssen wir nämlich daraus schliessen, dass $0 \leqq G - K \leqq \epsilon$, also da ϵ beliebig ist, dass $G = K$ ist.

Um nun den Beweis des Eingangs der Nr. stehenden Satzes zu liefern, lassen wir zunächst $k = 2$ sein. Der Bereich (1) wird jetzt durch das Quadrat zwischen den Absscissen $x_1 = a$ und $x_1 = \beta$ und den Ordinaten $x_2 = a$, $x_2 = \beta$ dargestellt. Umgeben wir jeden der Punkte $x_1 = \gamma_1, \cdots, x_1 = \gamma_m$ auf der Axe OX_1 mit einer Strecke von der Länge δ, so bleiben von der Strecke $a\beta$ $m + 1$ Theile übrig, wovon keiner einen der soeben erwähnten Punkte enthält. Das-

selbe werde auf der Axe OX_2 vorgenommen. Ziehen wir von den auf diese Weise markirten $2m$ Punkten der Strecke $a\beta$ auf der Axe OX_1 Parallele zur Axe OX_2, und von den $2m$ auf der Axe OX_2 markirten Punkten Parallele zur Axe OX_1, so werden uns dem vorhin genannten Quadrate $(m+1)^2$ Rechtecke ausgeschieden, deren sämmtliche Punkte von $\gamma_1 \cdots \gamma_m$ verschiedene Coordinaten besitzen. Für alle Punkte eines solchen Rechteckes ist demnach die Function $f(x_1, x_2)$ stetig. Somit gehört zu $\epsilon' > 0$ ein $\Delta' > 0$ so dass

$$0 \leqq \sum_r (g_r - k_r)\delta_r < \epsilon'$$

ist, wenn nur die Theile $\delta_1 \delta_2 \cdots$, in welche jede Seite dieses Rechteckes zerlegt ist, sämmtlich kleiner als Δ' sind. Wiederholen wir das an jedem der obigen $(m+1)^2$ Rechtecke, bezeichnen die Theile, in welche die Strecke $\beta - a$ auf die beschriebene Weise getheilt wird, mit $\delta_1 \cdots \delta_n$ und bemerken schliesslich, dass überhaupt $g_r - k_r \leqq B - A$ ist, so finden wir, dass

$$0 \leqq \sum_{r=1}^{n} (g_r - k_r)\delta_r < (m+1)\epsilon' + m(B-A)\delta$$

ist. Bei der Willkürlichkeit von ϵ' und δ stellt uns die rechte Seite dieser Beziehung jede beliebige positive Zahl ϵ vor. Also ist bei der angegebenen Wahl der Theile $\delta_1 \cdots \delta_n$

$$\sum_{r=1}^{n} (g_r - k_r)\delta_r < \epsilon.$$

Damit ist der in Rede stehende Satz für den Fall $k = 2$ gezeigt. Der Beweis für den Fall dass $k > 2$ ist gestaltet sich ganz ähnlich. Jetzt werden aus dem Bereiche (1) $(m+1)^k$ Bereiche ausgeschieden, in ihren jedem alle Stellen von $\gamma_1 \cdots \gamma_m$ verschiedene Coordinaten haben, sodoss $f(x_1 \cdots x_k)$ in jedem solchen Theilbereiche ausnahmslos stetig ist.

10. Wir setzen nun wieder wie in Nr. 6

$$f(x_1, x_2) = \sqrt{\phi'(x_1)^2 + \psi'(x_2)^2}.$$

Um diese Function auch für solche Punkte zu erklären, von deren Coordinaten mindestens eine einen der Werthe $\gamma_1 \cdots \gamma_m$ annimmt, setzen wir fest, dass falls $\phi(t)$ für $t = \gamma_r$ keinen vollständigen Differentialquotienten besitzt, $\phi'(\gamma_r)$ einen mittleren Werth zwischen den Grenzwerthen $\phi'(\gamma_r - 0)$ und $\phi'(\gamma_r + 0)$ bedeute und unter der nämlichen Voraussetzung Aehnliches bezüglich $\psi'(\gamma_r)$ gelte. Da die Functionen $|\phi'(t)|$ und $|\psi'(t)|$ im Intervalle (a, β) von t endliche obere Grenzen g' h' haben sollen, so ist die Function $f(x_1, x_2)$ in dem Quadrate zwischen den Abscissen $x_1 = a$, $x_2 = \beta$ und den Ordinaten $x_2 = a$, $x_2 = \beta$ endlich. In den Punkten, von deren Coordinaten mindestens eine einer der Zahlen $\gamma_1 \cdots \gamma_m$

gleich ist, ist $f(x_1, x_2)$ unstetig. Somit existirt nach No. 9 lim $\sum_r f_r \delta_r$ für lim $\delta_r = 0 (r = 1, \cdots, n)$ und zwar ist nach Nr. 4

$$\mathop{\mathrm{L}}_{\delta_r=0} \sum_{r=1}^{n} f_r \delta_r = \int_\alpha^\beta \sqrt{\phi'(t)^2 + \psi'(t)^2}\, dt.$$

Setzen wir für die Zahlen $f'_s (s = 1, \cdots, m)$ in Nr. 8 die Grenzen 0 und $\sqrt{g'^2 + h'^2}$ fest, so haben wir auch

$$\mathop{\mathrm{L}}_{\delta_r=0} \sum_{r=1}^{n} f'_r \delta_r = \int_\alpha^\beta \sqrt{\phi'(t)^2 + \psi'(t)^2}\, dt.$$

Die Summe auf der linken Seite umfasst zufolge der Formel (9) auch die Summe $\sum_r |A_{r-1}, A_r|$ für den in Rede stehenden Fall. Es ist mithin die Formel (8) in dem Sinne erwiesen, dass *den Theilen* $\delta_1 \cdots \delta_n$ *der Strecke* $\beta - a$ *keine Beschränkung auferlegt zu werden braucht.*

11. *Das Integral einer Function von* $z = x + yi$ *auf dem Wege* $\mathfrak{w}$ *mit der analytischen Darstellung*

$$z = g(t) = \phi(t) + \psi(t) i \qquad (a \leqq t \leqq \beta). \tag{13}$$

C. JORDAN hat bemerkt,* dass eine in allen Punkten eines gegebenen Weges längs derselben eindeutige und stetige Function $f(x)$ auf ihm integrirbar ist, *wenn er rectificirbar ist.* Haben die Functionen $\phi(t)$ und $\psi(t)$ die ihnen entweder in No. 6 oder in Nr. 7 beigelegten Eigenschaften, so existirt demnach

$$\int_{(\mathfrak{w})} f(z)\, dz. \tag{14}$$

Im ersteren Falle, dass $\phi'(t)$, $\psi'(t)$ im Intervalle (a, β) von t ausnahmlos stetig sind, kann man dieses Integral leicht in eines nach der reellen Veränderlichen t verwandeln.† Und zwar ist

$$\int_{(\mathfrak{w})} f(z)\, dz = \int_\alpha^\beta f\{g(t)\} g'(t)\, dt. \tag{15}$$

Um diese Formel auch für den zweiten Fall, in welchem der Weg $\mathfrak{w}$, den $m + 1$ Intervallen $(a, \gamma_1) \cdots (\gamma_m, \beta)$ von t entsprechend, in $m + 1$ Stücke $\mathfrak{v}_0$ $\mathfrak{v}_1 \cdots \mathfrak{v}_m$ von der nämlichen Beschaffenheit, wie der ganze Weg $\mathfrak{w}$ in dem ersten Falle, zerfällt, zu beweisen,‡ braucht man nur zu bemerken, dass

* *Cours d'Analyse*, 2. éd., I, Nr. 193.

† Vgl. meine *Grundzüge*, II, S. 174.

‡ Der hierfür von mir in den Monatsheften für Mathematik und Physik, Bd. XI, S. 64 gegebene Beweis befriedigt aus einem ähnlichen Grunde nicht ganz, wie die Beziehung (7) in Nr. 2 nicht ausreicht zur Rechtfertigung der Behauptung, dass das Integral (2) (S. 24), die in der Erklärung (D) an die Zahl L gerichtete Forderung erfülle.

$$\int_{(w)} f(z)\,dz = \sum_{r=0}^{m} \int_{(w_r)} f(z)\,dz,$$

ist und auf jedes der Integrale rechts die Formel (15) anzuwenden.

12. Die Rectification der durch die Gleichungen

$$x = \phi(t), \quad y = \psi(t), \quad z = \chi(t) \qquad (\alpha \leqq t \leqq \beta)$$

erklärten räumlichen *Curve* kann unter der Voraussetzung, dass diese drei Functionen von t ähnliche Bedingungen, wie die beiden ersteren in Nr. 6 und 7 erfüllen, in der nämlichen Weise durchgeführt werden, wie die der ebenen Curve (a). Hierbei hat man in den Sätzen der Nrn. 4 und 7 $k = 3$ zu setzen.

13. Das im Vorstehenden benutzte Verfahren lässt sich auch auf die Complanation der krummen Flächen ausdehnen, wie wir noch kurz andeuten wollen.

Erklärung des Inhaltes eines begrenzten krummen Flächenstückes $\mathfrak{F}$. Die rechtwinkligen Coordinaten $x\,y\,z$ eines beliebigen Punktes desselben seien als Functionen zweier Parameter $u\,v$ dargestellt:

$$x = \phi(u, v), \quad y = \psi(u, v), \quad z = \chi(u, v).$$

Dabei soll dadurch, dass den Veränderlichen $u\,v$ ein endliches stetiges Gebiet ϕ in der uv-Ebene angewiesen wird, der Punkt xyz die ganze Fläche $\mathfrak{F}$ einmal beschreiben. Die Kürze wegen nehmen wir an, dass das Gebiet ϕ nur einen und zwar einfachen Rand besitze. Die äussersten Werthe der Absscissen der Randpunkte seien α, α', die äussersten Werthe ihrer Ordinaten β, β'.

Wir theilen die Strecken $\alpha' - \alpha$, $\beta' - \beta$ auf den Axen $v = 0$, $u = 0$ in beliebig viele Theile:

$$\alpha' - \alpha = \delta_1 + \delta_2 + \cdots + \delta_m, \quad \beta' - \beta = \epsilon_1 + \epsilon_2 + \cdots + \epsilon_n,$$

und ziehen durch die Theilpunkte der ersteren, deren Absscissen

$$\alpha_1 = \alpha + \delta_1, \quad \alpha_2 = \alpha_1 + \delta_2 \cdots \alpha' = \alpha_m = \alpha_{m-1} + \delta_m$$

seien, die Parallelen zur v-Axe, durch die Theilpunkte der letzteren, deren Ordinaten

$$\beta_1 = \beta + \epsilon_1, \quad \beta_2 = \beta_1 + \epsilon_2, \quad \cdots, \quad \beta' = \beta_n = \beta_{n-1} + \epsilon_n$$

seien, die Parallelen zur u-Axe, wodurch das Gebiet ϕ mit einem Netz von Rechtecken $\delta_r \epsilon_s$ überzogen wird. Wir setzen

$$\phi(\alpha_r, \beta_s) = x_{rs}, \quad \psi(\alpha_r, \beta_s) = y_{rs}, \quad \chi(\alpha_r, \beta_s) = z_{rs},$$

und nennen den Punkt von $\mathfrak{F}$ mit diesen Coordinaten $M_{r,s}$.

Jedem Rechtecke in der uv-Ebene mit den Ecken

$$(1) \qquad (\alpha_{r-1}, \beta_{s-1}) \quad (\alpha_{r-1}, \beta_s) \quad (\alpha_r, \beta_s) \quad (\alpha_r, \beta_{s-1}),$$

welches bloss aus Punkten von ϕ besteht, entspricht ein windschiefes Viereck, dessen Ecken die Punkte $M_{r-1,s-1}\ M_{r-1,s}\ M_{r,s}\ M_{r,s-1}$ von $\mathfrak{F}$ sind. Wir bilden die Summe der Dreiecksflächen

$$(2) \qquad |M_{r-1,s-1}\, M_{r-1,s}\, M_{r,s-1}| + |M_{r,s-1}\, M_{r-1,s}\, M_{r,s}|$$

und addiren alle diese Summen, welche den bloss aus Punkten von ϕ bestehenden Rechtecken (1) entsprechen. Wenn die so erhaltene Summe, die wir mit S bezeichnen, bei unbeschränkter und unbegränzter Abnahme der Theile δ_r $(r = 1, \cdots, m)$ und ϵ_s $(s = 1, \cdots, n)$, deren Summen jedoch bezw. $a' - a$ und $\beta' - \beta$ bleiben müssen, einen endlichen Grenzwerth A besitzt, so nennen wir ihn die *Zahl der Fläche* $\mathfrak{F}$ d. h. *es soll jeder positiven Zahl* κ *eine andere solche Zahl* Δ *so entsprechen, dass wenn nur ein jedes* δ_r *und ein jedes* ϵ_s *kleiner als* κ *ist, alsdann stets*

$$(3) \qquad |S - \mathrm{A}| < \kappa$$

ausfällt.

14. Nun mögen die Functionen ϕ, ψ, χ bei jedem Punkte von ϕ stetig sein und zunächst dasselbe auch gelten von allen ihren partiellen Differentialquotienten nach u und v:

$$\frac{\partial \phi}{\partial u} = \phi'_u(u, v), \quad \frac{\partial \phi}{\partial v} = \phi'_v(u, v), \text{ u. s. w.}$$

Setzen wir sodann

$$\frac{\partial \psi}{\partial u}\frac{\partial \chi}{\partial v} - \frac{\partial \psi}{\partial v}\frac{\partial \chi}{\partial u} = J, \quad \frac{\partial \chi}{\partial u}\frac{\partial \phi}{\partial v} - \frac{\partial \chi}{\partial v}\frac{\partial \phi}{\partial u} = K, \quad \frac{\partial \phi}{\partial u}\frac{\partial \psi}{\partial v} - \frac{\partial \phi}{\partial v}\frac{\partial \psi}{\partial u} = L,$$

so ist *die Zahl der Fläche* $\mathfrak{F}$ *das Doppelintegral*

$$(4) \qquad \int\int_{(\phi)} \sqrt{J^2 + K^2 + L^2}\, du\, dv.$$

Um diesen Satz zu zeigen, berechne man zuerst die Summe (2). Verstehen wir unter $M\ M'\ M''\ M'''$ bezw. die den Werthsystemen

$$(u, v), \quad (u, v + \epsilon), \quad (u + \delta, v), \quad (u + \delta, v + \epsilon),$$

entsprechenden Punkte von $\mathfrak{F}$ und setzen

$$\psi'_u(u'', v)\, \chi'_v(u, v''') - \psi'_v(u, v'')\, \chi'_u(u''', v) = J_1,$$
$$\chi'_u(u''', v)\, \phi'_v(u, v') - \chi'_v(u, v''')\, \phi'_u(u', v) = K_1,$$
$$\phi'_u(u', v)\, \psi'_v(u, v'') - \phi'_v(u, v')\, \psi'_u(u'', v) = L_1,$$

und ähnlich

$$\psi'_u(u''_1, v+\epsilon)\chi'_v(u+\delta, v'''_1) - \psi'_v(u+\delta, v''_1)\chi'_u(u'''_1, v+\epsilon) = J_2,$$

u. s. w., so ist

$$(5) \quad |MM'M''| + |M''M'M'''| = \tfrac{1}{2}\{\sqrt{J_1^2 + K_1^2 + L_1^2} + \sqrt{J_2^2 + K_2^2 + L_2^2}\}\,\delta\epsilon.$$

Dabei bedeuten $u'\ u''\ u'''\ u'_1\ u''_1\ u'''_1$ nicht näher bekannte Werthe zwischen u und $u+\delta$, $v'\ v''$ u. s. w. eben solche Werthe zwischen v und $v+\epsilon$. In (5) hat man, um die Zahl (2) zu erhalten, $u = a_{r-1}$, $v = \beta_{s-1}$, $\delta = \delta_r$, $\epsilon = \epsilon_s$ zu setzen.

Dass dann die oben eingeführte Summe S bei $\lim \delta_r = 0$ $(r = 1, \cdots, m)$ und $\lim \epsilon_s = 0$ $(s = 1, \cdots, n)$ das Doppelintegral (4) zum Grenzwerthe hat, kann mit Hilfe eines Satzes nachgewiesen werden, welcher als eine Verallgemeinung des Satzes in Nr. 4 erscheint. Er lautet:

„Es sei

$$(6) \qquad f(x_1 \cdots x_k, y_1 \cdots y_k)$$

ein für jedes System $x_1 \cdots x_k, y_1 \cdots y_k$, wovon jedes Paar $x_1y_1 \cdots x_k y_k$ dem Bereiche ϕ angehört, eindeutige und für diesen Bereich endliche Function. Versteht man dann unter $f_{r,s}$ einen beliebigen Werth im Intervalle, welches gebildet wird von der oberen und unteren Grenze der Function (6), während $x_1y_1 \cdots x_ky_k$ beliebige Punkte des Rechtecks (1) sein dürfen, und hat die Summe $\sum f_{r,s}\delta_r\epsilon_s$, erstreckt über alle Rechtecke (1), welche bloss aus Punkten von ϕ bestehen, bei $\lim \delta_r = 0$ $(r = 1, \cdots, m)$ und $\lim \epsilon_s = 0$ $(s = 1, \cdots, n)$ einen endlichen Grenzwerth J, so lässt die Function

$$f(u \cdots u, u \cdots v)$$

ein eigentliches Doppelintegral über das Gebiet ϕ zu und zwar ist

$$\int\int_{(\phi)} f(u \cdots u, v \cdots v)\,du\,dv = J.$$"

Durch eine ähnliche Betrachtung wie in Nr. 5 lässt sich zeigen, dass die Summe $\sum f_{r,s}\delta_r\epsilon_s$ bei $\lim \delta_r = 0$ und $\lim \epsilon_s = 0$ einen endlichen Grenzwerth besitzt, wenn die Function (6) an allen Stellen $x_1 \cdots x_k, y_1 \cdots y_k$, wobei ein jedes Paar $x_1y_1 \cdots x_ky_k$ einen Punkt von ϕ bezeichnet, stetig ist.

Aus dem Vorstehenden ergiebt sich endlich die Existenz des in Nr. 13 erwähnten Grenzwerthes von S indem man als Function (6) den Ausdruck

$$\sqrt{J_1^2 + K_1^2 + L_1^2}$$

wählt, darin $u\ u'\ u''\ u'''$ bezw. durch $x_1x_2x_3x_4$, $v\,v'\,v''\,v'''$ bezw. durch $y_1y_2y_3y_4$ ersetzend. Alsdann wird

$$f(u \cdots u, v \cdots v) = \sqrt{J^2 + K^2 + L^2}.$$

15. Der in der letzten Nr. angeführte Satz wird durch Ueberlegungen ähnlicher Art, wie sie in Nr. 8 und 9 angestellt wurden, ausgedehnt auf den Fall dass *die sechs partiellen Differentialquotienten* $\partial\phi/\partial u$, $\partial\phi/\partial v$ u. s. w. *im Gebiete* ϕ *endlich und in allen Punkten desselben mit Ausnahme einer endlichen Anzahl von einzelnen Punkten und derer einer endlichen Anzahl von gewöhnlichen* * *Linien stetig sind.* Der hier angedeutete Beweis des solchergestalt erweiterter Satzes ist meines Erachtens allein völlig befriedigend.

* Vgl. meine *Grundzüge*, III, S. 37.

THE GROUPS OF STEINER IN PROBLEMS OF CONTACT*

BY

LEONARD EUGENE DICKSON

1. The problems of contact discussed by STEINER† and HESSE† were investigated from a more general standpoint by CLEBSCH in his paper on the application of Abelian functions to geometry.‡ A study of the groups of these geometrical problems has been made by JORDAN.§ One of the most interesting of these groups was shown by JORDAN to be holoedrically isomorphic with the first hypoabelian linear group, which plays so important a rôle in various geometrical questions and in the problem of the construction of all solvable groups. As the proof (*Traité*, pp. 229–249) is quite complicated, it seemed to the writer worth while to publish the elementary proof given below of the isomorphism in question. No use will be made of the JORDAN substitutions $[\alpha_1, \beta_1, \cdots, \alpha_p, \beta_p]$, neither the origin nor the interpretation of which is apparent.

2. The theorem that there are 28 bitangents to a curve of the fourth order has been generalized by CLEBSCH (l. c., § 8) as follows: Let C_n be a curve of order n having no double points and set $p = \frac{1}{2}(n-1)(n-2)$. *There are* $2^{p-1}(2^p - 1)$ *curves of order* $n-3$ *having simple contact with* C_n *at* $\frac{1}{2}n(n-3)$ *points.* The determination of these curves depends upon an equation E of degree $R_p \equiv 2^{2p-1} - 2^{p-1}$, whose roots may be represented by the symbol $(x_1 y_1 \cdots x_p y_p)$, where $x_1, y_1, \cdots, x_p, y_p$ may be 0 or 1, such that

$$x_1 y_1 + x_2 y_2 + \cdots + x_p y_p \equiv 1 \pmod 2. \tag{1}$$

Let μ be any integer, $\mu \gtrless R_p$, such that $\mu(n-3)/2$ is also an integer, and consider the μ roots

$$(x_1' y_1' \cdots x_p' y_p'), \cdots, (x_1^{(\mu)} y_1^{(\mu)} \cdots x_p^{(\mu)} y_p^{(\mu)}).$$

CLEBSCH proved that the points of contact of C_n with the corresponding μ

* Presented to the Society (Chicago) April 6, 1901, in connection with a paper entitled "Representation of linear groups as transitive substitution groups." Received for publication May 4, 1901.

† Journal für Mathematik, vol. 49 (1855).

‡ Ibid., vol. 63 (1864), pp. 189–243.

§ *Traité des substitutions*, pp. 329–333, 305–308, 229–249.

curves all lie on a curve of order $\mu(n-3)2$, if the following congruences hold simultaneously:

$$(2)\quad x_i' + x_i'' + \cdots + x_i^{(\mu)} \equiv 0, \quad y_i' + y_i'' + \cdots + y_i^{(\mu)} \equiv 0 \pmod 2 \quad (i = 1, \cdots, p).$$

Let ϕ_μ denote the sum of the products of the R_p roots taken μ at a time. According to a general principle,* the substitutions of the group G of the equation E will leave the function ϕ_μ invariant. If n be even, μ can have only even values, so that G is a subgroup of the group† which leaves $\phi_4, \phi_6, \cdots$ invariant. If n be odd, μ can be any integer such that $2 < \mu \leqq R_p$, and the group G is contained in the group G_1' defined by the invariants $\phi_3, \phi_4, \cdots, \phi_{R_p}$. We are to prove that G_1' is holoedrically isomorphic with the first hypoabelian group G_0 on $2p$ indices with coefficients taken modulo 2.

3. The first hypoabelian group G_0 is formed by the substitutions

$$S: \qquad \xi_i' = \sum_{j=1}^{p}(a_{ij}\xi_j + \gamma_{ij}\eta_j), \quad \eta_i' = \sum_{j=1}^{p}(\beta_{ij}\xi_j + \delta_{ij}\eta_j) \quad (i = 1, \cdots, p),$$

with coefficients taken modulo 2, which leave formally invariant the function

$$\theta \equiv \xi_1\eta_1 + \xi_2\eta_2 + \cdots + \xi_p\eta_p.$$

As generators of G_0, we may take

$$(3)\qquad M_i \equiv (\xi_i\eta_i), \quad N_{ij}: \ \xi_i' = \xi_i + \eta_j, \quad \xi_j' = \xi_j + \eta_i,$$

where we have written only the indices altered by the substitution.

The substitution S replaces the function

$$f = \sum_{i=1}^{p}(x_i\xi_i + y_i\eta_i)$$

by

$$f' = \sum_{i=1}^{p}(x_i'\xi_i + y_i'\eta_i), \quad x_i' \equiv \sum_{j=1}^{p}(a_{ji}x_j + \beta_{ji}y_j), \quad y_i' \equiv \sum_{j=1}^{p}(\gamma_{ji}x_j + \delta_{ji}y_j).$$

The x_i', y_i' are expressed in terms of x_j, y_j by formulæ which define a matrix of coefficients identical with the transposed of the matrix of the coefficients of S. Hence these formulæ define a substitution of the group G_0 (as shown by the explicit conditions on the coefficients of a first hypoabelian substitution).‡ Hence

$$(4)\qquad x_1'y_1' + x_2'y_2' + \cdots + x_p'y_p' = x_1y_1 + x_2y_2 + \cdots + x_py_p.$$

*Compare JORDAN, *Traité*, no. 421.

†Shown by JORDAN, nos. 319–335, to be holoedrically isomorphic with the Abelian linear group on $2p$ indices with coefficients taken modulo 2.

‡Cf. Bulletin of the American Mathematical Society, vol. 4 (1898), pp. 495–510.

This result may also be shown by considering the generators (3). In fact, M_1 and N_{12} replace the function f by, respectively,

$$y_1\xi_1 + x_1\eta_1 + \sum_{i=2}^{p}(x_i\xi_i + y_i\eta_i),$$

$$x_1\xi_1 + (y_1 + x_2)\eta_1 + x_2\xi_2 + (y_2 + x_1)\eta_2 + \sum_{j=3}^{p}(x_i\xi_i + y_i\eta_i).$$

In view of (4), it follows that S permutes amongst themselves the functions f in which $x_1y_1 + \cdots + x_py_p = 1$. In place of the functions f, we may employ the positional symbols $(x_1y_1\cdots x_py_p)$ of § 2. Hence G_0 is isomorphic with a substitution-group Γ on these R_p symbols. Moreover, the isomorphism is holoedric and the group Γ is transitive; these results are readily proved.*

4. We may write the functions ϕ_3 and ϕ_4 as follows:

$$\phi_3 = \sum(x_1'y_1'\cdots x_p'y_p')(x_1''y_1''\cdots x_p''y_p'')(x_1' + x_1''\ y_1' + y_1''\cdots x_p' + x_p''\ y_p' + y_p''),$$
$$\phi_4 = \sum(x_1'y_1'\cdots)(x_1''y_1''\cdots)(x_1'''y_1'''\cdots)(x_1' + x_1'' + x_1'''\ y_1' + y_1'' + y_1'''\cdots),$$

the summations extending over all the symbols $(x_1'y_1'\cdots)$, $(x_1''y_1''\cdots)$, $(x_1'''y_1'''\cdots)$, such that the final term is, in each case, a symbol. Thus, for ϕ_3,

$$\sum_{i=1}^{p}x_i'y_i' \equiv 1, \quad \sum_{i=1}^{p}x_i''y_i'' \equiv 1, \quad \sum_{i=1}^{p}(x_i' + x_i'')(y_i' + y_i'') \equiv 1 \pmod 2.$$

Let G_1 be the group of STEINER composed of all substitutions on the R_p symbols $(x_1y_1\cdots x_py_p)$ which leave ϕ_3 and ϕ_4 invariant.† We first show that G_1 contains the group Γ as a subgroup. In fact, M_1 replaces the general term (written above) of ϕ_3 by

$$(y_1'x_1'\ x_2'y_2'\cdots)(y_1''x_1''\ x_2''y_2''\cdots)(y_1' + y_1''\ x_1' + x_1''\ x_2' + x_2''\ y_2' + y_2''\cdots),$$

which is also a term of ϕ_3. Similarly, M_i and N_{ij} leave ϕ_3 and ϕ_4 invariant. Hence G_1 contains all the generators of Γ. The next step consists in the proof that every substitution of G_1 belongs to Γ. From the two results we may then conclude that $G_1 \equiv \Gamma$, so that G_1 and the first hypoabelian group G_0 will be proved holoedrically isomorphic.

5. Let L be an arbitrary substitution of G_1 and denote by f_1 the symbol which L replaces by $(00\ 11\ 00\cdots 00)$. Then Γ, being transitive, contains a substitution L' which replaces f_1 by $(00\ 11\cdots 00)$. Hence $M \equiv L'^{-1}L$ will belong to G_1 and will leave $(00\ 11\cdots 00)$ fixed. Since M does not alter ϕ_3, it

* American Journal of Mathematics, vol. 23 (1901), pp. 337–377, § 26.

† It appears in the sequel that $G_1 \equiv G_1'$, the latter (§2) leaving $\phi_3, \phi_4, \cdots, \phi_{R_p}$ invariant.

will leave invariant the function ϕ_3' given by the sum of those terms in ϕ_3 which contain the factor $(00\ 11\ \cdots\ 00)$:

$$\phi_3' \equiv \sum (00\ 11\ 00\ \cdots)(x_1y_1\ x_2y_2\ x_3y_3\cdots)(x_1y_1\ x_2 + 1\, y_2 + 1\ x_3y_3\cdots),$$

$$(5) \qquad \sum_{i=1}^{p} x_iy_i \equiv 1, \quad \sum_{i=1}^{p}{}' x_iy_i + (x_2+1)(y_2+1) \equiv 1 \pmod 2,$$

where the accent denotes that the value $i = 2$ is excluded. Hence $x_2 + y_2 \equiv 1 \pmod 2$. Note that every set of solutions of (5) makes the three symbols in every triple of ϕ_3' all different. Hence M leaves invariant

$$\psi \equiv \sum (x_1y_1\ x_2y_2\ x_3y_3\cdots)(x_1y_1\ x_2 + 1\, y_2 + 1\ x_3y_3\cdots).$$

6. Hence M permutes amongst themselves the N symbols * *not* contained in the function ψ and different from $(00\ 11\ 00\ \cdots)$, namely, the symbols $(x_1y_1\ x_2y_2\cdots)$ for which, in contrast to (5),

$$(6) \qquad \sum_{i=1}^{p} x_iy_i \equiv 1, \quad \sum_{i=1}^{p} x_iy_i + x_2 + y_2 + 1 \equiv 0 \pmod 2.$$

Hence $x_2 + y_2 \equiv 0 \pmod 2$. We next prove that the substitutions of Γ which leave fixed $(00\ 11\ \cdots)$ permute transitively the N symbols defined by (6). Among them occurs $(10\ 11\ 00\ \cdots\ 00)$. We are to prove that Γ contains a substitution Σ leaving fixed the symbol $(00\ 11\ \cdots\ 00)$ and replacing $(10\ 11\ \cdots\ 00)$ by an arbitrary symbol $(x_1y_1\cdots x_py_p)$ in which

$$(6') \qquad \sum_{i=1}^{p} x_iy_i \equiv 1, \quad x_2 \equiv y_2 \pmod 2.$$

In view of § 3, we may think of the literal substitutions of Γ as linear hypoabelian substitutions on $\xi_1, \eta_1, \cdots \xi_p, \eta_p$. We are therefore to prove that there exists a first hypoabelian substitution S which leaves $\xi_2 + \eta_2$ fixed and replaces $\xi_1 + \xi_2 + \eta_2$ by

$$\sum_{i=1}^{p} (x_i\xi_i + y_i\eta_i),$$

subject to (6'). Hence S must leave $\xi_2 + \eta_2$ fixed and replace ξ_1 by

$$\sum_{i=1}^{p} (x_i\xi_i + y_i\eta_i) - \xi_2 - \eta_2, \quad \sum_{i=1}^{p}{}' x_iy_i + (x_2-1)(y_2-1) \equiv 0, \quad x_2 \equiv y_2 \pmod 2.$$

* The number $N = R_p - 2R_{p-1} - 1 \equiv 2^{2p-2} - 1$. Indeed, the number of sets of solutions of (5) equals the number of sets of solutions of

$$\sum_{i=1}^{p}{}' x_iy_i \equiv 1,$$

since $(x_2+1)x_2$ is even, which number is evidently R_{p-1}.

7. Changing the notation, we are to prove that G_0 contains a substitution S_1 which leaves $\xi_1 + \eta_1$ fixed and replaces ξ_2 by

$$\sum_{i=1}^{p}(X_i\xi_i + Y_i\eta_i), \quad \sum_{i=1}^{p} X_iY_i \equiv 0, \quad X_1 \equiv Y_1 \pmod 2.$$

If $X_1 \equiv 0$, we take as S_1 a substitution * which leaves ξ_1 and η_1 fixed and replaces ξ_2 by

$$\sum_{i=2}^{p}(X_i\xi_i + Y_i\eta_i), \quad \sum_{i=2}^{p} X_iY_i \equiv 0 \pmod 2.$$

If $X_1 \not\equiv 0 \pmod 2$, then $X_2, Y_2, \cdots, X_p, Y_p$ are not all zero. Applying a suitable transformation on $\xi_2, \cdots, \xi_p$, we may suppose that $X_2 \neq 0$. Now G_0 contains the following substitution leaving $\xi_1 + \eta_1$ invariant:

$$W \equiv Q_{21a}N_{12a}: \begin{cases} \xi_1' = \xi_1 + a\eta_2, & \eta_1' = \eta_1 + a\eta_2, \\ \xi_2' = \xi_2 + a\xi_1 + a\eta_1 + a^2\eta_2, & \eta_2' = \eta_2. \end{cases}$$

Also G_0 contains the substitution V which leaves ξ_1 and η_1 fixed and replaces ξ_2 by

$$X_2\xi_2 + (Y_2 + X_1^2/X_2)\eta_2 + \sum_{i=3}^{p}(X_i\xi_i + Y_i\eta_i),$$

since, in view of $X_1 = Y_1$,

$$X_2(Y_2 + X_1^2/X_2) + \sum_{i=3}^{p} X_iY_i = X_1^2 + \sum_{i=2}^{p} X_iY_i = \sum_{i=1}^{p} X_iY_i = 0.$$

If we take $a = X_1/X_2$, the required substitution S_1 is the product WV.

8. It follows that $M = \Sigma P$, where P is a substitution of G_1 which leaves fixed the symbols (00 11 00 $\cdots$ 00) and (10 11 00 $\cdots$ 00). Hence P leaves invariant the two functions ϕ_3' and ϕ_3'' formed respectively by those terms of ϕ_3 which contain (00 11 $\cdots$) and (10 11 $\cdots$) as a factor. Hence P leaves invariant the function ψ of § 5 derived from ϕ_3', and the following function derived from ϕ_3'':

$$\psi_1 \equiv \Sigma(x_1y_1\,x_2y_2\,x_3y_3\cdots)(x_1 + 1\,y_1\,x_2 + 1\,y_2 + 1\,x_3y_3\cdots).$$

Hence P will permute amongst themselves the symbols occurring in ψ_1 and not in ψ. These symbols $(x_1y_1\cdots)$ are defined by

$$\sum_{i=1}^{p} x_iy_i \equiv 1, \quad \sum_{i=1}^{p}{}' x_iy_i + (x_2 + 1)(y_2 + 1) \equiv 0,$$

$$(x_1 + 1)y_1 + (x_2 + 1)(y_2 + 1) + \sum_{i=3}^{p} x_iy_i \equiv 1 \pmod 2.$$

* Bulletin, l. c., p. 498.

Hence there are 2^{2p-3} such symbols satisfying the conditions

$$\sum_{i=1}^{p} x_i y_i \equiv 1, \quad x_2 \equiv y_2, \quad y_1 \equiv 1 \pmod 2. \tag{7}$$

Among them occurs (01 11 00 $\cdots$ 00). We next prove that Γ contains a substitution T which leaves fixed (00 11 $\cdots$), (10 11 $\cdots$) and replaces (01 11 $\cdots$) by an arbitrary symbol ($x_1 y_1\, x_2 y_2 \cdots$) satisfying the conditions (7). We are to find a substitution T of the first hypoabelian group G_0 which leaves fixed $\xi_2 + \eta_2$ and $\xi_1 + \xi_2 + \eta_2$, but replaces $\eta_1 + \xi_2 + \eta_2$ by $x_1\xi_1 + y_1\eta_1 + x_2\xi_2 + y_2\eta_2 + \cdots$ subject to the relations (7). Then T must leave fixed ξ_1 and $\xi_2 + \eta_2$, but replace η_1 by

$$x_1\xi_1 + \eta_1 + (x_2+1)\xi_2 + (x_2+1)\eta_2 + \sum_{i=3}^{p}(x_i\xi_i + y_i\eta_i) \quad \Big[x_1 + x_2 + \sum_{i=3}^{p} x_i y_i \equiv 1\Big].$$

Such a substitution belonging to G_0 is the following:

	ξ_1	η_1	ξ_2	η_2	ξ_3	η_3	$\cdots$	ξ_p	η_p
$\xi_1' =$	1	0	0	0	0	0	$\cdots$	0	0
$\eta_1' =$	x_1	1	x_2+1	x_2+1	x_3	y_3	$\cdots$	x_p	y_p
$\xi_2' =$	x_2+1	0	1	0	0	0	$\cdots$	0	0
$\eta_2' =$	x_2+1	0	0	1	0	0	$\cdots$	0	0
$\xi_3' =$	a_{31}	γ_{31}	a_{32}	γ_{32}	a_{33}	γ_{33}	$\cdots$	a_{3p}	γ_{3p}
	.	.	.	.	.	.	.	.	.
$\eta_p' =$	β_{p1}	δ_{p1}	β_{p2}	δ_{p2}	β_{p3}	δ_{p3}	$\cdots$	β_{pp}	δ_{pp}

Since the coefficients of the first four rows satisfy the first hypoabelian conditions which affect those rows, there exist values of

$$a_{ij},\ \gamma_{ij},\ \beta_{ij},\ \delta_{ij} \qquad (i = 3, \cdots, p\,;\ j = 1, \cdots, p)$$

for the remaining $2p - 4$ rows which give rise to a first hypoabelian substitution.*

9. It follows that $P = TQ$, where Q is a substitution of G_1 which leaves fixed the symbols (00 11 00 $\cdots$), (10 11 00 $\cdots$), and (01 11 00 $\cdots$). Since Q leaves ϕ_4 fixed, it will leave invariant the functions τ, τ_1, τ_2 which occur in ϕ_4 each multiplied by the respective factors (00 11 $\cdots$)(10 11 $\cdots$), (00 11 $\cdots$)(01 11 $\cdots$), (10 11 $\cdots$)(01 11 $\cdots$), namely,

$$\tau \equiv \sum (x_1 y_1\, x_2 y_2 \cdots)(x_1 + 1\ y_1\ x_2 y_2 \cdots),$$
$$\tau_1 \equiv \sum (x_1 y_1\, x_2 y_2 \cdots)(x_1 y_1 + 1\ x_2 y_2 \cdots),$$
$$\tau_2 \equiv \sum (x_1 y_1\, x_2 y_2 \cdots)(x_1 + 1\ y_1 + 1\ x_2 y_2 \cdots).$$

**The successive generality theorem*, American Journal, l. c.

Hence Q will permute amongst themselves the q symbols which occur in τ and τ_1, but not in τ_2, subject therefore to the conditions

$$\sum_{i=1}^{p} x_i y_i \equiv 1, \quad (x_1+1)y_1 + \sum_{i=2}^{p} x_i y_i \equiv 1, \quad x_1(y_1+1) + \sum_{i=2}^{p} x_i y_i \equiv 1,$$

$$(x_1+1)(y_1+1) + \sum_{i=2}^{p} x_i y_i \equiv 0 \pmod 2.$$

Hence

$$x_1 \equiv y_1, \quad x_1 y_1 \equiv 0, \quad \sum_{i=2}^{p} x_i y_i \equiv 1 \pmod 2.$$

We obtain the q symbols*

$$(8) \qquad (00\, x_2 y_2\, x_3 y_3 \cdots), \quad \sum_{i=2}^{p} x_i y_i \equiv 1.$$

10. The theorem that $G_1 = \Gamma$ may now be proved by induction from $2(p-1)$ to $2p$ indices. We denote by $\phi_3^{(p-1)}, \phi_4^{(p-1)}, \cdots$ the functions composed of those terms of $\phi_3, \phi_4, \cdots$, respectively, which are formed exclusively of the symbols $(00\, x_2 y_2\, x_3 y_3 \cdots)$. We assume that every substitution which leaves fixed $\phi_3^{(p-1)}, \phi_4^{(p-1)}$ is derived from the substitutions of Γ_{p-1}, the first hypoabelian group on $p-1$ pairs of indices; and proceed to prove that every substitution which leaves fixed ϕ_3, ϕ_4 is derived from the substitutions of $\Gamma \equiv \Gamma_p$. In view of the earlier sections we need only consider the substitutions of the form Q which permute amongst themselves the q symbols (8). Let Q' be the substitution derived from Q by retaining only the cycles on the q symbols. Since Q' leaves $\phi_3^{(p-1)}$ and $\phi_4^{(p-1)}$ invariant, it belongs to Γ_{p-1} by hypothesis. We proceed to show that $K \equiv QQ'^{-1}$ reduces to the identity, so that the theorem will be proved. Now K leaves fixed every symbol $(00\, x_2 y_2 x_3 x_3 \cdots)$, as well as $(01\ 11\ 00 \cdots)$, and $(10\ 11\ 00 \cdots)$. Hence it leaves fixed the fourth term of ϕ_4 in the products

$$(00\ 11\ 00 \cdots)(10\ 11\ 00 \cdots)(00\ x_2 y_2\, x_3 y_3 \cdots),$$
$$(00\ 11\ 00 \cdots)(01\ 11\ 00 \cdots)(00\ x_2 y_2\, x_3 y_3 \cdots),$$

which are $(10\, x_2 y_2\, x_3 y_3 \cdots)$ and $(01\, x_2 y_2\, x_3 y_3 \cdots)$, respectively. Hence K leaves fixed the fourth term of ϕ_4 in the product

$$(10\ 1 + x_2 + x_3 y_3\ 0\ 11\ 00 \cdots)(01\ 0\ y_2 + 1\ 11\ 00 \cdots)(00\ 1 + x_3 y_3\ 1\ x_3 y_3\, x_4 y_4 \cdots),$$

which is $\sigma \equiv (11\, x_2 y_2 x_3 y_3\, x_4 y_4 \cdots)$, where $x_4 y_4 + \cdots + x_p y_p \equiv 0$, $x_2 y_2 + x_3 y_3 \equiv 0$. But, in every symbol σ, $x_2 y_2 + x_3 y_3 + x_4 y_4 + \cdots + x_p y_p \equiv 0$. If there are any terms $\neq 0$, say $x_r y_r$ and $x_s y_s$, where $r > 1$, $s > 1$, $r \neq s$, then $x_r y_r + x_s y_s \equiv 0$

* Evidently, $q = R_{p-1} \equiv 2^{2p-3} - 2^{p-2}$.

(mod 2). Such a symbol σ may be reached in a manner analogous to that by which was obtained the σ having $x_2y_2 + x_3y_3 \equiv 0$.

Since K leaves fixed every symbol of the forms

$$(00\ x_2y_2\ x_3y_3\cdots),\quad (10\ x_2y_2\ x_3y_3\cdots),\quad (01\ x_2y_2\ x_3y_3\cdots),\quad (11\ x_2y_2\ x_3y_3\cdots),$$

it leaves every symbol fixed and is the identity.

11. The order Ω_p of the group $G_1 \equiv \Gamma$ may be derived from the preceding investigation. We have, for $p > 2$,

$$\Omega_p = R_p N 2^{2p-3}\Omega_{p-1} \div q \equiv (2^{2p-1} - 2^{p-1})(2^{2p-2} - 1)2^{2p-3}\Omega_{p-1} \div (2^{2p-3} - 2^{p-2}),$$

upon substituting the values of R_p, N and q given in the notes to § 6 and § 9. The factor $\Omega_{p-1} \div q$ expresses the number of substitutions on the q symbols $(00\ x_2y_2\ x_3y_3\cdots)$ which leave invariant the symbol $(00\ 11\ 00\cdots)$. But Γ_{p-1} is transitive on these q symbols.

After simplification, we derive the recursion formula

$$\Omega_p = 2^{2p-2}(2^p - 1)(2^{p-1} + 1)\Omega_{p-1} \qquad (p > 2).$$

The formula holds also for $p = 2$, if we take $\Omega_1 = 2$, as must be done in the case of Γ, the hypoabelian substitutions on ξ_1 and η_1 being M_1 and the identity. The definition of G_1 for $p = 1$ is delusive since $R_1 = 1$; but, for $p = 2$, G_1 is formed of the $36 \equiv 2(3!)^2$ substitutions on $R_2 = 6$ symbols which leave invariant

$$\phi_3 \equiv (00\ 11)(11\ 01)(11\ 10) + (11\ 00)(10\ 11)(01\ 11).$$

We readily find that *

$$\Omega_p = (2^p - 1)[(2^{2p-2} - 1)2^{2p-2}][(2^{2p-4} - 1)2^{2p-4}]\cdots[(2^2 - 1)2^2]2.$$

The factors of composition of Γ are known to be 2 and $\frac{1}{2}\Omega_p$, if $p > 2$.

12. The question of the generalization of the results of the paper from the field of integers taken modulo 2 to the Galois Field of order 2^m will be reserved for a later paper. It may be remarked that the results of §§ 3, 4, 5, and 7 are true for the $GF[2^m]$; but that the methods of §§ 6, 8, 9, and 10 would require essential modifications.

THE UNIVERSITY OF CHICAGO, *April*, 1901.

* This result is in accord with that obtained otherwise in the Bulletin, l. c.

QUATERNION SPACE*

BY

ARTHUR S. HATHAWAY

INTRODUCTION.

The recent paper by STRINGHAM in these Transactions (vol. 2, p. 183) has suggested the preparation of a consecutive development of my methods of treating quaternion space as they finally took form in the last of the series of my papers to which STRINGHAM refers; particularly since those papers are published in different journals, and for the most part in briefest abstract.

While our results have many points of contact with each other and with the results of previous authors, as is shown by STRINGHAM'S very thorough compilation of literature on the subject, yet our methods and points of view are entirely different. Also some results, particularly as to the quaternion form of the most general rigid displacement, were obtained by STRINGHAM long before I turned my attention to the subject.

STRINGHAM deals analytically with the equations of loci, and develops the geometry by the interpretation of those equations. I use a more synthetic method, which interprets the quaternion symbols themselves instead of the equations between them. This divergence between the analytic and the synthetic methods of employing the quaternion analysis constitutes the general difference between the methods of CAYLEY and of TAIT.

CLIFFORD has stated this synthetic view in his *Further Note on Biquaternions* (*Mathematical Papers*). CLIFFORD makes, however, an artificial limitation in that paper when he says that "we may regard the *last* symbol of a product as either a concrete number or a symbol of operation; but all others must be regarded as symbols of operation." I held this view in my first paper (Bulletin, Nov., 1897) and was forced by it to the development of two systems of quaternions (the direct and the contra systems). Immediately after the publication of that paper, I discovered that both systems became one and the same system, by the abandonment of the restriction that only the *last* symbol could be concrete.

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STRINGHAM points out that my theory of contra and direct parallel great circles in the hypersphere of fourfold space corresponds exactly to CLIFFORD's theory of right and left parallels in elliptic space, in which "great circles" correspond to "straight lines," and "great spheres" to "planes." I may say, further, that CLIFFORD's right and left vector translations correspond to my contra and direct turns; his motor twist, to my contra-direct turn. CLIFFORD's *right* and *left* refer to right-hand and left-hand screw movements; my *contra* and *direct* refer to the position of the operating symbol on *the right* or on *the left* of the concrete symbol. These positions of the operating factor (of which that on the right is not recognized by CLIFFORD) correspond to operative powers that are equivalent to the screw movements of CLIFFORD. I shall, therefore, hereafter use the terms *right* and *left* instead of *contra* and *direct*, to denote these positions of the multiplier.

COMPLEX NUMBER SPACE.

1. This preliminary discussion of n-fold space is made for the sake of completeness of development and point of view. Most of the results, and probably all, have been obtained by the usual analytical treatment of equations of loci. The method of dealing synthetically with complex numbers as the concrete elements corresponding to geometric elements, without reference to equations of loci, may possibly be new. There is no assumption that the geometric terms *point*, *direction*, *line*, *locus*, *etc.*, refer to anything else than complex numbers. Nevertheless, the results given will be sufficient to show that an exact correspondence may be developed between any threefold space and euclidean space. Proofs are omitted, because they depend only upon the simplest principles of the linear composition of independent complex units.

2. The space considered is the totality of points

$$i_1x_1 + i_2x_2 + \cdots + i_nx_n,$$

where the x's are variable *real* numbers, and the i's are given *independent* complex units, so that two points

$$i_1x_1' + \cdots + i_nx_n', \quad i_1x_1'' + \cdots + i_nx_n''$$

are the same only if

$$x_1' = x_1'', \cdots, x_n' = x_n''.$$

3. A contained m-fold ($m < n$) is a locus

$$a + a_1x_1 + a_2x_2 + \cdots + a_mx_m,$$

where $a_1, a_2, \cdots, a_m$ are independent complex numbers and a is any complex number. The *directions* of this locus are the differences (displacements) between its points, viz.,

$$a_1x_1 + a_2x_2 + \cdots + a_mx_m.$$

In particular the direction from a to b is the difference $b - a$. The context must show whether a given complex number is taken as a point or as a direction (or as a line as in art. 9).

Two directions, a_1, a_1x_1 are *like* directions in the same or in opposite *senses* according as x_1 is positive or negative.

4. A set of points and directions is an *independent* set when the directions from one point to each of the others form with the given directions an independent set of complex numbers. This definition applies to a set of points alone and to a set of directions alone.

5. The *elements* of an m-fold are its points and directions. An m-fold is determined by any set of $m + 1$ independent elements, of which one at least is a point. Directions alone cannot determine a space.

6. The *minimum space* of a given k-fold and l-fold is the m-fold of smallest number m which contains the given spaces. The *convergence* of the given spaces is the number c of their independent *common* elements and is given by the formula

$$c = k + l + 1 - m.$$

The *divergence* of the given spaces is the number d of independent elements besides the common elements which are necessary to determine the lesser space. If $k \gtrless l$, we have

$$d = k + 1 - c = m - l.$$

7. Two spaces of convergence c intersect, if at all, in the $(c - 1)$-fold determined by the c independent common elements. When the common elements are all directions, the two spaces do not intersect. For example, two planes (two-folds) of minimum four-fold space have $c = 1$; i. e., they intersect in one point with no common direction, or they have one direction in common and do not intersect.

8. *Parallels.* Two spaces are *parallel* when one space contains all the directions of the other. Hence

If two parallel spaces intersect, then one space contains the other.

Two distinct parallel spaces are spaces of unit divergence, which do not intersect.

For example, parallel straight lines (one-folds) are straight lines which are coplanar and non-intersecting.

Through a given point one and only one k-fold can be drawn parallel to a given k-fold; and two k-folds parallel to the same k-fold are parallel to each other.

9. *Lines.* The locus

$$a + xp \qquad (0 \leqq x \leqq 1)$$

is a segment of the straight line $a + xp$ between a and $a + p$. Considered as generated from a to $a + p$ by increasing x, its direction is p; its *length* is

$$Tp = \surd(x_1^2 + x_2^2 + \cdots + x_n^2),$$

where

$$p = i_1 x_1 + i_2 x_2 + \cdots + i_n x_n.$$

This segment considered as to length and direction will be called *the line p;* considered also as to initial point, it is *the line p drawn from a*.

10. Lines are added according to the displacement law $AB + BC = AC$, viz.,

$$(b - a) + (c - b) = (c - a).$$

Also, since $T(2p) = 2Tp$, etc., we have ordinary principles of measurement upon a straight line; while the inequality $T(p + q) \leqq Tp + Tq$ determines principles of shortest geometric distance.

11. *Angles.* The angle between the directions (or lines) p and q will be, concretely, the locus of directions

$$p \cos \phi + q \sin \phi \qquad (0 \leqq \phi \leqq \pi/2).$$

The magnitude of this angle, in notation $\angle(p, q)$, will be defined so that we have euclidean relations in the parallelogram whose sides are p, q and diagonal $p + q$, viz.,

$$T(p + q)^2 = Tp^2 + Tq^2 + 2TpTq \cos \angle(p, q).$$

If $p = i_1 x_1 + i_2 x_2 + \cdots$, $q = i_1 y_1 + i_2 y_2 + \cdots$, this formula gives

$$\cos \angle(p, q) = \frac{(x_1 y_1 + x_2 y_2 + \cdots)}{Tp \cdot Tq}, \quad = \frac{S \cdot pKq}{Tp \cdot Tq},$$

by an extension of quaternion notation.

12. HAMILTON calls the angle θ (between 0 and π) whose cosine is Sq/Tq, *the angle of q;* and calls the unit vector $\alpha = UVq$, *the axis of q*. In notation

$$Uq = \cos \theta + \alpha \sin \theta = e^{\alpha\theta}.$$

The above definition of angle becomes then in quaternions

$$\angle(p, q) = \angle pKq = \angle pq^{-1} = \cdots;$$

so that in particular, *two lines are perpendicular when their quotient is a vector.*

13. The function

$$S \cdot pKq = x_1 y_1 + x_2 y_2 + \cdots + x_n y_n,$$

whose vanishing is the condition of perpendicularity of the lines or directions p, q, is linear and symmetric in these complex variables, and in consequence we easily find:

A direction perpendicular to each of several directions is perpendicular to all of their linear compositions.

A set of mutually perpendicular directions is an independent set; and any m-fold contains any number of sets of m determining directions which are thus mutually perpendicular. The locus of directions perpendicular to all the directions of a given r-fold will be the directions of a certain $(n-r)$-fold.

14. *Orthogonals.* An r-fold and an $(n-r)$-fold such that every direction of one is perpendicular to every direction of the other will be called *orthogonal spaces.* The convergence of such orthogonal spaces is

$$c = r + (n-r) + 1 - n = 1.$$

Hence,

Orthogonal spaces intersect in one and only one point; and through a given point one and only one space can be drawn orthogonal to a given space.

Two spaces orthogonal to the same space are parallel to each other.

15. The orthogonal projection of a point upon a given space is therefore a definite point; and the line joining the point with its projection is the *unique* perpendicular from the point to the given space, and the *shortest line* from the given point to the given space.

16. *The orthogonal projection of a line upon a given space is another line generated by the orthogonal projection of the generating point of the given line; and such projection is one of the two components of the line in the given and an orthogonal space.*

17. *Transformations.* The transposition of the points of one figure into the points of another is a *transformation.* The transformation is *continuous* when the transposition takes place by a continuous and simultaneous change of value of all numbers of the figure from their initial to their final values. A continuous transformation is *rigid*, when it leaves invariant the distances between all pairs of points. In consequence, straight lines remain straight lines, planes remain planes, etc., in every rigid transformation, because shortest distances remain such; also the magnitudes of angles are unaltered, and in particular all perpendicular and orthogonal relations, and relative senses* of arrangement of points, lines, etc., are unchanged, because triangles, tetrahedrons, etc., remain rigid.

18. *Translations.* The addition of a given number to all points of a figure does not change distances or directions between points; and this will be called a rigid *translation* of the *magnitude*, *direction*, and *sense* of the given number.

19. *Multiplications.* The multiplication of all points of a given figure on the left or right by a given number p is a transformation which depends upon the multiplication table. The point 0 is an invariant point or *center.* The left multi-

* This part is rigorously an extension of a threefold conception rather than a theorem.

plication p, followed by the translation $a - pa$ which together leave the point a invariant, will be called *the left multiplication p about the center a;* in this transformation the point r becomes the point $a + p(r - a)$.

20. *Dilations.* A right or left multiplication p may be separated into its tensor and versor parts Tp, Up, which are interchangeable. The tensor multiplication, Tp, increases all lengths in the ratio $Tp : 1$, and leaves directions and their senses unaltered. It is *a dilation about the center, of magnitude Tp.*

Quaternion Space.

21. *Right and Left Turns.* A right or left versor multiplication may be called a *right or left turn about the center.* It is in the quaternion analysis, to which we now confine ourselves, *a rigid displacement*, on account of the tensor property $T \cdot pq = Tp \cdot Tq$; for $T(pr - ps) = T \cdot p(r - s) = T(r - s)$, since $Tp = 1$. Also, the transformation is made continuous by continuous variation of p from 1 to its final value.

In what follows, results are *dual* with respect to the terms *right* and *left*; i. e., one should repeat all statements and proofs which are given with the interchange of *right* and *left* terms and significations.

22. *A left turn of given axis a and variable angle θ has for its invariant linear spaces a certain system of planes through the center such that one and only one invariant plane contains a given direction. The angular displacement in every invariant plane is of magnitude θ in a fixed sense determined by the axis; and the path curves of points are circles in these planes about the center.* (See art. 12 in connection with this article.)

For since $\angle(r, qr) = \angle qrr^{-1} = \angle q$, therefore by a left turn q every direction r receives the angular displacement $\angle q = \theta$. There can therefore be no invariant lines (except for the special values $\theta = 0$ or π). Hence there can be no invariant spaces not through the center; for if there were such a space then the unique perpendicular upon it from the center would be an invariant line. Similarly, there can be no invariant threefold through the center, since a threefold has an unique orthogonal line at the center. It remains to consider planes through the center.

If an invariant plane through the center contain a direction r then it must contain the direction $e^{a\theta}r$, and such a plane will therefore contain the direction ar as one of the linear compositions of $e^{a\theta}r = r \cos\theta + ar \sin\theta$, with r; so that such a plane can only be the plane (r, ar) through the center. Further, any direction of this plane is $e^{a\phi}r$, which transforms into the direction $e^{a\theta}e^{a\phi}r = e^{a(\phi+\theta)}r$ of the same plane by an angular displacement θ in the sense of the plane (from r to ar). Finally, the path of a point in an invariant plane (from the invariance of distances) is at a constant distance from the center, i. e., it is a circle about the center.

23. *A series of parallel planes of given angular sense are directionally invariant planes of a left turn of one and only one axis* a, *whose sense of turn is that of the planes. Such axis* a *will be called the left axis of each plane. Reversal of angular sense reverses the corresponding left axis.*

For, since $qp^{-1} \cdot p = q$,

$$a = UV \cdot qp^{-1}$$

is the axis of a left turn which leaves the plane of directions (p, q) invariant and displaces its directions in its angular sense (from p to q). If β be the axis of *any* such left turn, then we see, on remembering that the axis turns through a right angle in the sense of the turn, that

$$\beta p = ap \quad \text{or} \quad \beta = a.$$

For the reversed plane of directions (q, p), the corresponding left axis is

$$UV \cdot pq^{-1} = -\, UV \cdot qp^{-1} = -\, a.$$

(The unit vectors of reciprocal numbers are opposites.)

24. Two planes with the same or opposite left axes will be called *left parallel planes in the same or opposite senses.** From article 22 we have, *one and only one plane can be drawn through a given straight line left parallel to a given plane.*

Transferred to the hypersphere, which intersects lines and planes through its center in points and great circles, this result is: *One and only one great circle may be drawn through a given point left parallel to a given great circle.*

25. *There is one and only one plane of directions and only one angular sense, corresponding to given right and left axes, viz., such a plane must contain the vector direction that bisects the angle between the given axes.*

Let a, β be the given left and right axes, and r any direction of any plane corresponding to these axes; then we must have

$$ar = r\beta. \dagger$$

Since $a^2 = \beta^2 = -1$, an obvious solution is $r = a + \beta$, the vector bisector of the angle (a, β). (This is a vector δ perpendicular to a when $\beta = -a$.) Thus a required plane and sense is $(a + \beta,\ a\,a + \beta)$. There cannot be more than two independent solutions for r; for if there were three, r_1, r_2, r_3, then their linear composition would determine a threefold solution (r_1, r_2, r_3), and $a(r_1, r_2, r_3)$ would determine a fourth independent solution, since no threefold is invariant for left turn a. Thus all values of r would be solutions (in particular $r = \pm 1$), which is impossible.

* STRINGHAM calls these left (and also the right) parallel planes *isoclines*.

† This is STRINGHAM's equation of the plane in question, which passes through the origin; but without reference to angular sense.

26. Let $\{a, \beta\}$ signify a plane whose left axis is a, and right axis, β, viz., a plane $(a+\beta, a\overline{a+\beta})$. The opposite plane is then $\{-a, -\beta\}$; the *orthogonal plane* is $\{a, -\beta\}$ or $\{-a, \beta\}$ according to angular sense (since $a+\beta, a(a+\beta), (a-\beta), a(a-\beta)$ are mutually perpendicular). Also $\{a, a\} = (1, a)$ and $\{a, -a\} = (\delta, a\delta)$, a plane of vector directions perpendicular to a. To correspond with conventions of TAIT, the *left* turn a must be a *right handed* turn about the axis a in the plane of vector space which is perpendicular to a.

27. Two orthogonal planes are seen to be both right and left parallel planes; if right parallel in the same sense then they are left parallel in opposite senses.

All planes of left axis a are included in the form

$$(a) \qquad \{a, \pm\lambda\}$$

where λ is an arbitrary unit vector; from which it is seen that such planes occur in orthogonal pairs.

28. *A left turn of axis a and a right turn of axis β have one orthogonal pair of invariant planes in common, $\{a, \beta\}$ and $\{a, -\beta\}$ and only one* (art. 27); *in the plane $\{a, \beta\}$ the two turns are in the same sense (the sense of the plane); in the plane $\{a, -\beta\}$, the two turns are opposed in sense (the left turn with the sense of the plane).*

*The Double Türn $e^{a\theta}(\ )e^{\beta\phi}$.**

29. The invariant planes of the double turn of axes a, β are evidently the common orthogonal pair of invariant planes of the component turns, viz., $\{a, \beta\}$, $\{a, -\beta\}$. The path curves in these planes are circles about the center and the angular displacement in the first plane is $\theta+\phi$, and in the second it is $\theta-\phi$, in the senses of those planes (art. 28).

30. For a given point of initial position r outside an invariant plane a locus or path curve under the double turn $e^{a\theta}(\ )e^{\beta\phi}$ is

$$e^{a\theta} r e^{\beta\phi}$$

for a continuous simultaneous variation of θ, ϕ from 0, 0 to their final values. For independent variations of θ, ϕ, the above locus is an invariant surface, generated by all possible path curves of the point r, which we call *the hypercircular ring through the point r, of cyclic planes* $\{a, \beta\}$, $\{a, -\beta\}$.

31. The above ring is a surface in the threefold boundary of the hypersphere of radius Tr, with its center at the center of turning. The equation $\phi = \phi_1$,

* The *right* and *left turn*, in the hypersphere CLIFFORD'S *motor twist*. STRINGHAM calls it a *rotation*, because path curves in the invariant planes are circles. In the present development, which proceeds much farther than STRINGHAM'S, it appears that path curves are generally spirals, as in twists; whereas the path curves of rotations are concentric circles in each of a system of parallel planes.

where ϕ_1 is a constant, determines a great circle section of this ring with a left axis a, so that the family of curves $\phi = const.$ is a generating system of left parallel great circles. Similarly $\theta = \theta_1$ determines a great circle section of the ring with a right axis β; and the family $\theta = const.$ is a generating system of right parallel great circles. Two generating circles of the same system cannot intersect because they are right or left parallel; but two generating circles of opposite systems always intersect in the point $e^{a\theta}re^{\beta\phi}$ corresponding to the given values of θ, ϕ on each circle. The ring is ruled by its two systems of generating circles in "great arc parallelograms." (See art. 41.)

32. It will be proved in art. 35 that $e^{a\psi}(\ )e^{\beta\psi}$ is a rotation about the invariant plane $\{a, -\beta\}$ as a fixed axis. Thus we may interpret the identity:

$$e^{a\psi}(e^{a\theta}re^{\beta\phi})e^{\beta\psi} = e^{a(\theta+\psi)}re^{\beta(\phi+\psi)},$$

to mean that the ϕ generating circle of the left parallel system rotates about the axis of revolution into the $\phi + \psi$ generating circle of the same system. By changing $e^{\beta\psi}$ into $e^{-\beta\psi}$ we make $\{a, \beta\}$ the axis of revolution. Hence

Either generating system of circles of a hypercircular ring is a system of revolution about either cyclic plane of the ring as axis of revolution.

33. If r_1, r_2 be the orthogonal projections of a line (or point) r upon the cyclic planes of the above ring, then $r = r_1 + r_2$, and hence

$$e^{a\theta}re^{\beta\phi} = e^{a\theta}r_1e^{\beta\phi} + e^{a\theta}r_2e^{\beta\phi}.$$

This resolution of any point of the ring (or line from the center to that point) into components in the cyclic planes must therefore be its orthogonal projections on those planes; and, since the loci of these projections are circles about the center of radii Tr_1, Tr_2, *the orthogonal projection of a hypercircular ring on either cyclic plane is a circle about the center.*

34. The hypercircular ring is CLIFFORD'S *surface of zero curvature generated by a series of parallel lines meeting a fixed line.* The planes of the circles of either generating system of a hypercircular ring form what STRINGHAM calls either an *ordinal* or a *cardinal* system of planes.* There is only *one type* of linear one dimensional systems of right or left parallel planes (isoclines); and STRINGHAM'S division into two types, *ordinal* and *cardinal*, is a distinction founded on common properties of the one type. The ordinal property is that of article 31, the cardinal property is that of article 32. The properties of the hypercircular ring were presented by me in 1898 before the Chicago Section of the American Mathematical Society, and before the American Association as part of a complete development of the types of the invariant figures, path curves and surfaces, of all conformal transformations. †

*Transactions of the American Mathematical Society, vol. 2 (1901), p. 211.

†Bulletin of the American Mathematical Society, November, 1898, p. 93.

35. *Rotations.* The double turn $e^{\alpha\theta}(\ )e^{\beta\theta}$ is characterized by a *fixed* plane $\{\alpha, -\beta\}$ (displacement $\theta - \theta = 0$); while *any* orthogonal plane $\{\alpha, \beta\}$ is invariant, since it intersects the fixed plane in a fixed point. The path curves in each plane $\{\alpha, \beta\}$ are therefore circles about its fixed point as center; and the angular displacement is 2θ. This double turn is therefore *a rotation;* the fixed plane is the *plane axis;* the orthogonals to the fixed plane are *equators.*

36. A *left* rotation is one in which the angular sense of its plane axis is determined by its component left turn. The left rotation about the axis $(p, q) = \{\alpha, -\beta\}$ through twice the angle (p, q) is given by

$$qp^{-1}(\ )q^{-1}p = e^{\alpha\theta}(\ )e^{\beta\theta},$$

where

$$\alpha = UV\cdot qp^{-1}, \quad \beta = UV\cdot q^{-1}p, \quad \theta = \angle(p, q).$$

With the same axis and angle, the corresponding right rotation is

$$pq^{-1}(\ )p^{-1}q = e^{-\alpha\theta}(\ )e^{-\beta\theta}.$$

37. Taking $p = 1$ in the preceding rotations we obtain the left and right rotations $q(\ )q^{-1}$, $q^{-1}(\ )q$ about the axis $(1, q) = (1, Vq)$ through twice the angle $(1, q)$, which is HAMILTON's *twice the angle of* q. The equator lies in vector space, perpendicular to Vq; and vector space rotates within itself about Vq as axis. The left rotation is *right-handed* about the axis.

38. The resultant of two left rotations of vector space $q(\ )q^{-1}$ and $r(\ )r^{-1}$ is $rq(\ )q^{-1}r^{-1} = \bar{r}\bar{q}(\ )\bar{r}\bar{q}^{-1}$ a left rotation. It thus follows that any two equal angles in vector space, with a common vertex, may be brought into coincidence by a single left rotation.

Turning motion of planes.

39. In left turning a plane about a center, *the left axis of the plane is left-rotated in vector space about the axis of the turn through twice its angle, and the right axis of the plane remains fixed.*

For let (q, r) be any plane, which becomes the plane (pq, pr) by a left turn p; then the axes of the given plane are (art. 23)

$$\alpha = UV\cdot rq^{-1}, \quad \beta = UV\cdot q^{-1}r,$$

and the axes of the transformed plane are

$$\alpha' = UV\cdot prq^{-1}p^{-1} = pap^{-1}, \quad \beta' = UV\cdot q^{-1}r = \beta.$$

40. We see now in applying a left turn to any figure of lines and planes through the center that *all lines through the center describe equal left parallel plane angles in the same sense, while the successive positions of any plane are right parallel in the same sense.*

41. In the hypersphere, about the given center, the *left turn* becomes a *left parallel translation* of all points through equal arc distances in the same sense, while successive positions of any great circle are right parallel in the same sense. If we join by a great arc any point of a given great circle to any point of another right parallel great circle, the transversing arc may be taken as the path curve of a left turn which displaces one great circle into the other; and in this turn, all points of the turning circle describe great arcs that are equal and left parallel to the given path curve. In other words, two right parallel great circles are everywhere equally distant by left parallel arc measurement. Also, in this motion the translated great circle generates the right parallel system of a hypercircular ring, while its points describe the left parallel generating system. Hence if the entire ring be subjected to the left turn of its left generating system then its surface advances with the left parallel generating circles as path curves, and the right parallel generating circles as successive positions of any one.* This shows in particular, that opposite sides and angles of great arc parallelograms are equal, and that adjacent angles are supplementary. A ring is determined by any parallelogram; two adjacent sides determine two parallelograms and two rings, viz., if a be a vertex, and b, c adjacent vertices, the opposite vertex is $ba^{-1}c$ or $ca^{-1}b$.

42. We need not repeat the proof in the Bulletin for November, 1897 (p. 57fg.), that equidistant translation of all points of a rigid arc is either right or left parallel translation. It is not necessary for that proof that *all* points be given as equidistantly translated; it is sufficient if only *three points* of the arc be so translated, no two of them being diametrically opposite.

Angles between planes.

43. The angle between the left axes of two planes will be called their *left axial angle.* We have

The axial angles of two planes are unaltered in magnitude by the rigid displacements of single or double turns. (Art. 39.)

One pair of planes is rigidly reconcilable with another pair of planes, in repect to their directions and angular senses, when and only when the magnitudes of corresponding axial angles are equal. (Arts. 38, 39.)

The most general rigid displacement is the resultant of a double turn and a translation.

44. The *plane angle* between two planes which have a common direction is the angle whose sides are formed by turning such a common direction through a right angle in the sense of turn of each plane. Such planes through the center meet in a line, and this is the usual plane angle of their diedral angle.

* The left turn by which right parallel arcs are reconcilable is a particular right-handed twist; this displacement property is the foundation for CLIFFORD'S term *right-parallel lines.*

Planes may be separated by parallel translation of either so as to have only a common direction and no point of intersection, without altering their plane angle. In the hypersphere about the center, the great circles of diametric planes that intersect in a line will intersect in a point, and the sides of each plane angle will be tangential directions of the corresponding circles at their point of intersection, in the senses of the respective circles; and such plane angle will be *the angle between the great circles.*

45. *If two planes transverse (contain a common direction), then their axial angles are equal to each other and to their plane angle. Conversely, if the axial angles of two planes are equal to each other, then the planes transverse.*

For, let $\{a, \beta\}$ $\{a', \beta'\}$ be given planes which contain a common direction r; then their plane angle is

$$\angle(ar, a'r) = \angle a'r(ar)^{-1} = \angle a'a^{-1} = \angle(a, a') = \angle(r\beta, r\beta') = \angle(\beta, \beta').$$

Conversely, let $\angle(a, a') = \angle(\beta, \beta') = \theta$; then construct transversing planes with plane angle θ; their axial angles will each be θ, as just shown, and they will therefore be rigidly reconcilable in directions and senses with the given planes, by art. 43.

46. Two transversing planes are *perpendicular* when their plane angle is a right angle. *A common perpendicular to two planes* is a plane that transverses each perpendicularly.

47. *An orthogonal pair of common perpendiculars to two given planes can be drawn through a given point; and there is only one such pair if the given planes are neither right nor left parallel. If the given planes are left parallel there is a continuous system of right parallel orthogonal pairs of common perpendiculars.*

Let $\{a, \beta\}$, $\{a', \beta'\}$ be the given planes and let $\{a'', \beta''\}$ be any common perpendicular to them. Then

$$\angle(a, a'') = \angle(\beta, \beta'') = \pi/2, \quad \angle(a', a'') = \angle(\beta', \beta'') = \pi/2,$$

and therefore

$$a'' = \pm UVaa', \quad \beta'' = \pm UV \cdot \beta\beta',$$

which determines an unique orthogonal pair of common perpendiculars if a, a' are not parallel and β, β' are not parallel. If a, a' are parallel, then a'' may be any unit vector perpendicular to a, and to each of these values of a'' corresponds $\beta'' = \pm UV \cdot \beta\beta'$. If the given planes are orthogonal it will be seen that any common transversal is a common perpendicular to each.

48. *Plane angles.* For convenience consider planes meeting in a center so that transversing planes meet in a line through that point. All planes can be transferred to the center by parallel translation without altering direc-

tional relations. If we draw through the center an orthogonal pair of common perpendiculars to two given planes through the center we obtain a pair of intersections in each common perpendicular, whose included angle is a *plane angle* of the given planes. To be definite, let one plane be double turned about the orthogonal pair of common perpendicular planes as invariant planes into positional and sense coincidence with the other plane; then the intersections of the transferred plane with the invariant planes describe plane angles in the invariant planes; and it is these angles, so described, which are *the plane angles* of the two planes.

49. *The plane angles of two planes are the half sum and half difference of their axial angles.*

If $\{\alpha, \beta\}$, $\{\alpha', \beta'\}$ be the planes, so that $\theta = \angle(\alpha, \alpha')$, $\phi = \angle(\beta, \beta')$ are the axial angles, we take α'' perpendicular the plane of (α, α') on that side which left turns α into α', and take β'' perpendicular to the plane (β, β') on that side which right turns β into β'. The required double turn with the orthogonal pair of common perpendiculars $\{\alpha'', \pm\beta''\}$ as invariant planes, which brings the two planes into coincidence, is then $e^{\frac{1}{2}\alpha''\theta}(\ )\, e^{\frac{1}{2}\beta''\phi}$ (art. 39). Hence by art. 29, the required plane angles are $\frac{1}{2}(\theta + \phi)$, $\frac{1}{2}(\theta - \phi)$.

Orthogonal projection of areas.

50. The orthogonal projection of a line q on a plane $\{\alpha, \beta\}$ is a line $\frac{1}{2}(q - \alpha q \beta)$. For let $q = q_1 + q_2$, where $\alpha q_1 = q_1 \beta$, $\alpha q_2 = -q_2 \beta$, and we find $q_1 = \frac{1}{2}(q - \alpha q \beta)$.

51. The area of a parallelogram (p, q) will be

$$Tp \cdot Tq \cdot \sin \angle(p, q) = TV \cdot qKp.$$

In sign this parallelogram will be considered positive in the plane (p, q) and negative in the plane (q, p), so that it is the ratio of $V \cdot qKp$ to the left axis of the plane, $\pm UV \cdot qKp$.

52. *The area projecting factor between two planes is the product of the cosines of their plane angles.**

For if the area (p, q) be orthogonally projected from the plane

$$(p, q) = \{UV \cdot qKp,\ UV \cdot Kp \cdot q\} = \{\alpha', \beta'\}$$

upon the plane $\{\alpha, \beta\}$, the resulting area is

$$\tfrac{1}{4}\, V \cdot (q - \alpha q \beta) K(p - \alpha p \beta) \cdot \alpha^{-1} = -\tfrac{1}{2}\, S(\alpha\alpha' + \beta\beta') \cdot TV \cdot qKp.$$

Hence the projecting factor is

$$-\tfrac{1}{2}\, S(\alpha\alpha' + \beta\beta') = \tfrac{1}{2}(\cos\theta + \cos\phi) = \cos\tfrac{1}{2}(\theta + \phi) \cos\tfrac{1}{2}(\theta - \phi).$$

* STRINGHAM'S *measure of the divergence of the two planes.*

We have arrived at a convenient closing point so far as fundamental ideas are concerned. The treatment of conformal transformations would make a paper by itself and may be prepared later. I am not aware how much has been done in this direction in the ordinary theory of transformations of four real variables, but the quaternion analysis has certainly proven, as I view it, a powerful instrument for investigation in that direction. Also by its use, in connection with space conceptions, it may be said almost to take the place of fourfold eyes.

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RECIPROCAL SYSTEMS OF LINEAR DIFFERENTIAL EQUATIONS*

BY

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§ 1. *Dualistic geometrical interpretations of a system of two linear differential equations of the second order.*

The geometrical interpretation adopted in a former paper† was as follows. The system

$$\begin{aligned} y'' + p_{11}y' + p_{12}z' + q_{11}y + q_{12}z = 0\,, \\ z'' + p_{21}y' + p_{22}z' + q_{21}y + q_{22}z = 0\,, \end{aligned} \tag{1}$$

with the independent variable x was given. Its integration made known four pairs of independent simultaneous solutions

$$y_k = f_k(x), \quad z_k = g_k(x) \qquad (k = 1, 2, 3, 4). \tag{2}$$

The quantities (y_1, y_2, y_3, y_4) and (z_1, z_2, z_3, z_4) were interpreted as the homogeneous coördinates of two points P_y and P_z, so that equations (2) represented two curves in space, C_y and C_z. Corresponding points of the two curves C_y and C_z, *i. e.*, points belonging to the same value of x, were joined by a straight line L_{yz}. All of these straight lines generated a ruled surface S which was never a developable surface. The surface S was the same for all systems equivalent to (1), *i. e.*, for all systems which could be obtained from (1) by a transformation of the form

$$x = f(\bar{x}), \quad y = \alpha(\bar{x})\bar{y} + \beta(\bar{x})\bar{z}, \quad z = \gamma(\bar{x})\bar{y} + \delta(\bar{x})\bar{z}, \tag{3}$$

which transformation was moreover the most general point-transformation capable of transforming a general system of form (1) into another of the same form and order.‡

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† E. J. WILCZYNSKI, *Geometry of a simultaneous system of two linear homogeneous differential equations of the second order*, Transactions of the American Mathematical Society, vol. 2 (1901), no. 4.

‡ E. J. WILCZYNSKI, *Transformation of systems of linear differential equations*, American Journal of Mathematics, vol. 23 (1901).

Of course the following dualistic interpretation might also have been adopted. Let (y_k) and (z_k) be the coördinates of two planes, p_y and p_z. Then equations (2) are the equations of two developable surfaces D_y and D_z. Corresponding planes, *i. e.*, planes corresponding to the same value of x, will intersect in a straight line L_{yz}, and the totality of such lines will generate a ruled surface S, which again cannot be a developable surface. If x be left untransformed, the transformation (3) merely converts two planes p_y and p_z into two others $p_{\bar{y}}$ and $p_{\bar{z}}$ intersecting along the same line L_{yz}, while a transformation of x interchanges these lines in an arbitrary manner. The transformation (3), then, does not change the ruled surface S, which is therefore characteristic of a class of systems of differential equations equivalent by a transformation of the form (3).

Let us combine the two interpretations. At corresponding points P_y and P_z of the two fundamental curves C_y and C_z, let us construct the planes p_y and p_z tangent to the ruled surface S. These will intersect along the straight line L_{yz} which joins P_y and P_z. The four pairs of coördinates, determining these planes, will form a fundamental system of solutions for a new system of linear differential equations, as we shall show. This new system shall be called the *adjoined* of the original. Thus we shall have generalized the *Lagrange* equation adjoined to a single linear homogeneous differential equation.

§ 2. *Determination of the adjoined system.*

Let

$$D = \begin{vmatrix} y_1' & z_1' & y_1 & z_1 \\ y_2' & z_2' & y_2 & z_2 \\ y_3' & z_3' & y_3 & z_3 \\ y_4' & z_4' & y_4 & z_4 \end{vmatrix}, \tag{4}$$

so that

$$D = \sum_{k=1}^{4} y_k' v_k = -\sum_{k=1}^{4} z_k' u_k, \tag{5}$$

where

$$\begin{aligned} u_1 &= +(y_2' y_3 z_4), \quad u_2 = -(y_1' y_3 z_4), \quad u_3 = +(y_1' y_2 z_4), \quad u_4 = -(y_1' y_2 z_3), \\ v_1 &= +(z_2' y_3 z_4), \quad v_2 = -(z_1' y_3 z_4), \quad v_3 = +(z_1' y_2 z_4), \quad v_4 = -(z_1' y_2 z_3), \end{aligned} \tag{6}$$

the symbol

$$(a_i b_j c_k)$$

denoting a determinant of the third order whose main diagonal is $a_i b_j c_k$.

It is evident that the homogeneous coördinates of the planes tangent to the integrating ruled surface of system (1) at the points (y_i) and (z_i) respectively, are (u_i) and (v_i).

We shall now prove the following theorem: *If the two fundamental curves*

C_y and C_z, on the integrating ruled surface S, are transformed into two other curves $C_{\bar{y}}$ and $C_{\bar{z}}$ on the surface, by the transformation

$$y_k = a(x)\bar{y}_k + \beta(x)\bar{z}_k, \quad z_k = \gamma(x)\bar{y}_k + \delta(x)\bar{z}_k \qquad (k=1,2,3,4), \tag{7}$$

the developable surfaces formed by the planes tangent to the surface S along C_y and C_z are transformed by the equations

$$u_k = \Delta\,[a(x)\bar{u}_k + \beta(x)\bar{v}_k], \quad v_k = \Delta\,[\gamma(x)\bar{u}_k + \delta(x)\bar{v}_k] \qquad (k=1,2,3,4), \tag{8}$$

where

$$\Delta = a\delta - \beta\gamma, \tag{8a}$$

i. e., except for the factor Δ, by cogredient transformations.

Proof. We have, the summation everywhere being for $k=1,2,3,4$,

$$\begin{aligned} &\Sigma u_k z_k' = -D, \quad \Sigma u_k y_k' = 0, \quad \Sigma u_k z_k = 0, \quad \Sigma u_k y_k = 0, \\ &\Sigma v_k z_k' = 0, \qquad \Sigma v_k y_k' = D, \quad \Sigma v_k z_k = 0, \quad \Sigma v_k y_k = 0. \end{aligned} \tag{9}$$

Moreover it is clear that these eight equations are just sufficient to determine u_k and v_k. The values (7) of y_k and z_k being introduced, the following system, of which again u_k, v_k are the unique solutions, is obtained:

$$\begin{aligned} &\Sigma u_k(\gamma\bar{y}_k' + \delta\bar{z}_k' + \gamma'\bar{y}_k + \delta'\bar{z}_k) = -D, \quad \Sigma u_k(\gamma\bar{y}_k + \delta\bar{z}_k) = 0, \\ &\Sigma u_k(a\bar{y}_k' + \beta\bar{z}_k' + a'\bar{y}_k + \beta'\bar{z}_k) = 0, \qquad \Sigma u_k(a\bar{y}_k + \beta\bar{z}_k) = 0, \\ &\Sigma v_k(\gamma\bar{y}_k' + \delta\bar{z}_k' + \gamma'\bar{y}_k + \delta'\bar{z}_k) = 0, \qquad \Sigma v_k(\gamma\bar{y}_k + \delta\bar{z}_k) = 0, \\ &\Sigma v_k(a\bar{y}_k' + \beta\bar{z}_k' + a'\bar{y}_k + \beta'\bar{z}_k) = D, \qquad \Sigma v_k(a\bar{y}_k + \beta\bar{z}_k) = 0. \end{aligned} \tag{10}$$

By direct computation we find

$$D = \Delta^2\bar{D}. \tag{11}$$

Moreover we have the relations (9) between the transformed quantities $\bar{D}$, $\bar{u}_k$, $\bar{v}_k$, $\bar{y}_k$, $\bar{z}_k$, i. e.,

$$\begin{aligned} &\Sigma \bar{u}_k\bar{z}_k' = -\bar{D} = -D/\Delta^2, \quad \Sigma \bar{u}_k\bar{y}_k' = 0, \quad \Sigma \bar{u}_k\bar{z}_k = 0, \quad \Sigma \bar{u}_k\bar{y}_k = 0, \\ &\Sigma \bar{v}_k\bar{z}_k' = 0, \quad \Sigma \bar{v}_k\bar{y}_k' = \bar{D} = D/\Delta^2, \quad \Sigma \bar{v}_k\bar{z}_k = 0, \quad \Sigma \bar{v}_k\bar{y}_k = 0. \end{aligned} \tag{12}$$

Multiplying the first four equations of this set in order by δ, γ, δ', γ' and adding, we get the first equation of the following system:

$$\begin{aligned} &\Sigma \bar{u}_k(\gamma\bar{y}_k' + \delta\bar{z}_k' + \gamma'\bar{y}_k + \delta'\bar{z}_k) = -\delta D/\Delta^2, \quad \Sigma \bar{u}_k(\gamma\bar{y}_k + \delta\bar{z}_k) = 0, \\ &\Sigma \bar{u}_k(a\bar{y}_k' + \beta\bar{z}_k' + a'\bar{y}_k + \beta'\bar{z}_k) = -\beta D/\Delta^2, \quad \Sigma \bar{u}_k(a\bar{y}_k + \beta\bar{z}_k) = 0, \\ &\Sigma \bar{v}_k(\gamma\bar{y}_k' + \delta\bar{z}_k' + \gamma'\bar{y}_k + \delta'\bar{z}_k) = \gamma D/\Delta^2, \qquad \Sigma \bar{v}_k(\gamma\bar{y}_k + \delta\bar{z}_k) = 0, \\ &\Sigma \bar{v}_k(a\bar{y}_k' + \beta\bar{z}_k' + a'\bar{y}_k + \beta'\bar{z}_k) = aD/\Delta^2, \qquad \Sigma \bar{v}_k(a\bar{y}_k + \beta\bar{z}_k) = 0. \end{aligned} \tag{13}$$

But from (13) follows very easily a system of precisely the same form as (10), only with

$$\Delta(a\bar{u}_k + \beta\bar{v}_k) \quad \text{and} \quad \Delta(\gamma\bar{u}_k + \delta\bar{v}_k)$$

in place of u_k and v_k. But equations (10) were sufficient to determine u_k and v_k completely. Therefore we must have

$$u_k = \Delta(a\bar{u}_k + \beta\bar{v}_k), \quad v_k = \Delta(\gamma\bar{u}_k + \delta\bar{v}_k) \qquad (k=1, 2, 3, 4),$$

which proves the theorem.

If we put

$$U_k = \frac{u_k}{\sqrt{D}}, \quad V_k = \frac{v_k}{\sqrt{D}} \qquad (k=1, 2, 3, 4), \tag{14}$$

where the sign of the square root may be chosen at will, (U_k, V_k) are absolutely cogredient with (y_k, z_k).

For $x = \text{const.}$ we single out a generator of the ruled surface S and we consider two points P_y and P_z upon it, together with the planes tangent to S at these points. The transformation (7) then transforms P_y and P_z and their tangent planes into $P_{\bar{y}}$ and $P_{\bar{z}}$ and their tangent planes. The transformation (7) is now a linear transformation, and from the fact that the tangent planes are transformed by the cogredient transformations (8) follows the well-known theorem that *the anharmonic ratio of four points on any generator is the same as the anharmonic ratio of the corresponding tangent planes.*

If y_k and z_k form a fundamental system of solutions of equations (1) the determinant D does not vanish, i. e., the ruled surface S is a non-developable surface. If the corresponding determinant for u_k and v_k be formed, its value turns out to be D^3, and therefore also different from zero.

We can therefore regard u_k and v_k as constituting a simultaneous fundamental system of solutions of a pair of equations of the same form as (1). We proceed to set up this system of differential equations.

Denote the minors of z_k and y_k in D by ζ_k and η_k respectively, so that

$$D = \sum_{k=1}^{4} y_k \eta_k = \sum_{k=1}^{4} z_k \zeta_k, \tag{15}$$

where

$$\begin{aligned} &\eta_1 = +(y_2' z_3' z_4), \ \eta_2 = -(y_1' z_3' z_4), \ \eta_3 = +(y_1' z_2' z_4), \ \eta_4 = -(y_1' z_2' z_3), \\ &\zeta_1 = -(y_2' z_3' y_4), \ \zeta_2 = +(y_1' z_3' y_4), \ \zeta_3 = -(y_1' z_2' y_4), \ \zeta_4 = +(y_1' z_2' y_3). \end{aligned} \tag{16}$$

Then making use of (1) we shall find

$$\begin{aligned} u_k' &= \quad \zeta_k - p_{11} u_k - p_{12} v_k, \\ v_k' &= -\eta_k - p_{21} u_k - p_{22} v_k \end{aligned} \qquad (k=1, 2, 3, 4), \tag{17}$$

whence

$$
\begin{aligned}
u_k'' &= \zeta_k' - p_{11}\zeta_k + p_{12}\eta_k + (p_{11}^2 + p_{12}p_{21} - p_{11}')u_k \\
&\qquad + \{(p_{11}+p_{22})p_{12} - p_{12}'\}v_k, \\
v_k'' &= -\eta_k' - p_{21}\zeta_k + p_{22}\eta_k + \{(p_{11}+p_{22})p_{21} - p_{21}'\}u_k \\
&\qquad + (p_{22}^2 + p_{12}p_{21} - p_{22}')v_k.
\end{aligned} \tag{18}
$$

By differentiating (16) we find

$$
\eta_k' = -(p_{11}+p_{22})\eta_k + q_{11}v_k - q_{21}u_k,
$$

$$
\zeta_k' = -(p_{11}+p_{22})\zeta_k + q_{12}v_k - q_{22}u_k,
$$

and from (17)

$$
\eta_k = -v_k' - p_{21}u_k - p_{22}v_k,
$$

$$
\zeta_k = u_k' + p_{11}u_k + p_{12}v_k.
$$

Substituting these values in (18) we find that (u_k, v_k) *are simultaneous solutions of the following system of equations:*

$$
\begin{aligned}
&u'' + (2p_{11}+p_{22})u' + p_{12}v' + \{p_{11}' + q_{22} + (p_{11}+p_{22})p_{11}\}u \\
&\qquad + \{p_{12}' - q_{12} + (p_{11}+p_{22})p_{12}\}v = 0, \\
&v'' + p_{21}u' + (2p_{22}+p_{11})v' + \{p_{21}' - q_{21} + (p_{11}+p_{22})p_{21}\}u \\
&\qquad + \{p_{22}' + q_{11} + (p_{11}+p_{22})p_{22}\}v = 0.
\end{aligned} \tag{19}
$$

Moreover (u_k, v_k) *form a simultaneous fundamental system of* (19), since, as we have already seen, their determinant does not vanish.

We shall prefer, in general, to use another system, namely, that one whose solutions are the functions U_k and V_k as defined by equations (14). Remembering that

$$
D = Ce^{-\int(p_{11}+p_{22})dx}, \tag{20}
$$

one sees that we can obtain this other system from (19) by making the transformation

$$
u = Ue^{-\frac{1}{2}\int(p_{11}+p_{22})dx}, \quad v = Ve^{-\frac{1}{2}\int(p_{11}+p_{22})dx}.
$$

The resulting system is

$$
\begin{aligned}
&U'' + p_{11}U' + p_{12}V' + \{q_{11} + \tfrac{1}{4}(u_{11}-u_{22})\}U + (q_{12} + \tfrac{1}{2}u_{12})V = 0 \\
&V'' + p_{21}U' + p_{22}V' + (q_{21} + \tfrac{1}{2}u_{21})U + \{q_{22} + \tfrac{1}{4}(u_{22}-u_{11})\}V = 0,
\end{aligned} \tag{21}
$$

where u_{ik} are the same as the quantities so denoted in a previous paper.*

*E. J. WILCZYNSKI, *Invariants of systems of linear differential equations*, Transactions of the American Mathematical Society, vol. 2 (1901), no. 1, p. 6.

A third form, which may be convenient, is obtained by putting

$$u = \lambda e^{-\int (p_{11}+p_{22})dx}, \quad v = \mu e^{-\int (p_{11}+p_{22})dx}.$$

Its fundamental solutions are

$$\lambda_k = \frac{u_k}{D}, \quad \mu_k = \frac{v_k}{D} \qquad (k=1, 2, 3, 4).$$

The equations which λ and μ satisfy are

$$\begin{aligned} &\lambda'' - p_{22}\lambda' + p_{12}\mu' - (p'_{22} - q_{22})\lambda + (p'_{12} - q_{12})\mu = 0, \\ &\mu'' + p_{21}\lambda' - p_{11}\mu' + (p'_{21} - q_{21})\lambda - (p'_{11} - q_{11})\mu = 0. \end{aligned} \tag{22}$$

We shall speak of system (21) *as the system adjoined to* (1). Of course (19) and (22) have essentially the same properties as (21). But the relation of (21) to (1) is somewhat simpler because its solutions are absolutely cogredient with the solutions (1) under transformation (3), while those of (19) and (22) are cogredient with (1) except for a factor.

§3. *Properties of adjoined systems. Reciprocity.*

The relation of adjoined systems to each other is a very close one. In the first place *they have the same seminvariants and invariants.*

For, if we form the quantities u_{ik}, v_{ik}, w_{ik} for the system (21) and denote them by capital letters, we find

$$U_{11} = u_{22}, \quad U_{12} = -u_{12}, \quad U_{21} = -u_{21}, \quad U_{22} = u_{11}, \tag{23}$$

and similarly for V_{ik} and W_{ik}.

The relation between systems (1) *and* (21) *is a reciprocal one, i. e., if of two systems the second is the adjoined of the first, then the first is also the adjoined of the second.*

For let us denote the coefficients of (21) by P_{ik} and Q_{ik}. Then

$$\begin{aligned} &P_{ik} = p_{ik}, \\ &Q_{11} = q_{11} + \tfrac{1}{4}(u_{11} - u_{22}), \quad Q_{12} = q_{12} + \tfrac{1}{2}u_{12}, \\ &Q_{22} = q_{22} + \tfrac{1}{4}(u_{22} - u_{11}), \quad Q_{21} = q_{21} + \tfrac{1}{2}u_{21}. \end{aligned} \tag{24}$$

But this gives, on account of (23),

$$\begin{aligned} &p_{ik} = P_{ik}, \\ &q_{11} = Q_{11} + \tfrac{1}{4}(U_{11} - U_{22}), \quad q_{12} = Q_{12} + \tfrac{1}{2}U_{12}, \\ &q_{22} = Q_{22} + \tfrac{1}{4}(U_{22} - U_{11}), \quad q_{21} = Q_{21} + \tfrac{1}{2}U_{21}, \end{aligned}$$

i. e., p_{ik} and q_{ik} are formed from P_{ik} and Q_{ik} just as P_{ik} and Q_{ik} are formed from p_{ik} and q_{ik}. This proves the reciprocity of the two systems.

From (24) it will be noted that *the adjoined system coincides with the original, if and only if*

$$u_{11} - u_{22} = u_{12} = u_{21} = 0, \tag{25}$$

i. e., if and only if the integrating ruled surface is of the second order.*

This may also be shown very simply in another way. Since $-u_k$ and v_k are the minors of z_k' and y_k' respectively, in D, we have

$$\sum_{i=1}^{4} U_i y_i = 0, \quad \sum_{i=1}^{4} V_i z_i = 0, \quad \sum_{i=1}^{4} U_i z_i = 0, \quad \sum_{i=1}^{4} V_i y_i = 0. \tag{26}$$

But if the differential equations for U and V are the same as those for y and z, we must have

$$U_i = \sum_{k=1}^{4} c_{ik} y_k, \quad V_i = \sum_{k=1}^{4} c_{ik} z_k \qquad (k = 1, 2, 3, 4),$$

which, substituted in the preceding equations, show that the curves C_y and C_z are situated on the same quadratic surface, and that the line joining the points P_y and P_z is a generator of this surface.

The equations (25) form an invariant system. If then the two systems (21) and (1) are not identical, but merely equivalent, these equations (25) must also be fulfilled. We can therefore say more generally: *a system of two linear differential equations of the second order is equivalent to its adjoined system, if and only if its integrating ruled surface is of the second order.*

If the original system has either the semi-canonical or the canonical form the same is true of the adjoined system.

From our definitions of the quantities involved we have the following relations:

$$\begin{aligned} &\Sigma z_k' U_k = -\sqrt{D}, && \Sigma y_k' V_k = +\sqrt{D}, \\ &\Sigma y_k U_k = 0, && \Sigma y_k V_k = 0, \\ &\Sigma z_k U_k = 0, && \Sigma z_k V_k = 0, \\ &\Sigma y_k' U_k = 0, && \Sigma z_k' V_k = 0, \end{aligned} \tag{27}$$

where D is defined by equation (4) or more briefly,

$$D = D(y_k', z_k', y_k, z_k). \tag{28}$$

* E. J. WILCZYNSKI, *Geometry of a simultaneous system of two linear homogeneous differential equations of the second order*, Transactions of the American Mathematical Society, vol. 2 (1901), no. 4, equation (61).

It follows from the reciprocity of the two systems, and may also be verified directly by differentiation of (27) that

$$(29)\qquad \begin{aligned} &\Sigma V_k' y_k = -\sqrt{D}, && \Sigma U_k' z_k = +\sqrt{D},\\ &\Sigma U_k y_k = 0, && \Sigma U_k z_k = 0,\\ &\Sigma V_k y_k = 0, && \Sigma V_k z_k = 0,\\ &\Sigma U_k' y_k = 0, && \Sigma V_k' z_k = 0, \end{aligned}$$

where D in the first place stands for

$$D(U_k',\ V_k',\ U_k,\ V_k);$$

but this is the same as (28), for

$$(30)\qquad D(U_k',\ V_k',\ U_k,\ V_k) = D(y_k',\ z_k',\ y_k,\ z_k).$$

Now let y_k and z_k be simultaneously transformed into

$$(31)\qquad \bar{y}_k = \sum_{i=1}^{4} c_{ki} y_i, \quad \bar{z}_k = \sum_{k=1}^{4} c_{ki} z_i \qquad (k=1, 2, 3, 4),$$

where c_{ki} are constants, whose determinant does not vanish. We can look upon such a transformation as a change of the tetrahedron of reference.

Equations (27) will be satisfied by the transformed quantities as well. Therefore

$$\begin{aligned} &\sum_{i=1}^{4} z_i' \sum_{k=1}^{4} \frac{c_{ki}}{\sqrt{C}} \bar{U}_k = -\sqrt{D}, && \sum_{i=1}^{4} y_i' \sum_{k=1}^{4} \frac{c_{ki}}{\sqrt{C}} \bar{V}_k = +\sqrt{D},\\ &\sum_{i=1}^{4} y_i \sum_{k=1}^{4} \frac{c_{ki}}{\sqrt{C}} \bar{U}_k = 0, && \sum_{i=1}^{4} y_i \sum_{k=1}^{4} \frac{c_{ki}}{\sqrt{C}} \bar{V}_k = 0,\\ &\sum_{i=1}^{4} z_i \sum_{k=1}^{4} \frac{c_{ki}}{\sqrt{C}} \bar{U}_k = 0, && \sum_{i=1}^{4} z_i \sum_{k=1}^{4} \frac{c_{ki}}{\sqrt{C}} \bar{V}_k = 0,\\ &\sum_{i=1}^{4} y_i' \sum_{k=1}^{4} \frac{c_{ki}}{\sqrt{C}} \bar{U}_k = 0, && \sum_{i=1}^{4} z_i' \sum_{k=1}^{4} \frac{c_{ki}}{\sqrt{C}} \bar{V}_k = 0, \end{aligned}$$

where C denotes the determinant of the transformation (31).

But these are of exactly the same form as (27), except that U_k and V_k have been replaced by

$$\frac{1}{\sqrt{C}} \sum_{k=1}^{4} c_{ki} \bar{U}_k, \quad \frac{1}{\sqrt{C}} \sum_{k=1}^{4} c_{ki} \bar{V}_k$$

respectively. But on the other hand equations (27) are sufficient to determine U_k and V_k as their solutions.

Therefore we must have

$$U_i = \frac{1}{\sqrt{C}} \sum_{k=1}^{4} c_{ki} \overline{U}_k, \quad V_i = \frac{1}{\sqrt{C}} \sum_{k=1}^{4} c_{ki} \overline{V}_k \qquad (i = 1, 2, 3, 4), \tag{32}$$

or, solving for $\overline{U}_k$ and $\overline{V}_k$,

$$\overline{U}_k = \frac{1}{\sqrt{C}} \sum_{i=1}^{4} C_{ki} U_i, \quad \overline{V}_k = \frac{1}{\sqrt{C}} \sum_{i=1}^{4} C_{ki} V_i \qquad (k = 1, 2, 3, 4), \tag{33}$$

where C_{ki} is the minor of c_{ki} in the determinant

$$C = |c_{ki}|.$$

The sign of the square root must be so chosen that, for positive values of C, $\sqrt{C} > 0$. For, if (31) reduces to the identical substitution, (32) also must reduce to identity. The determinant of the substitution (32) is equal to C, as is that of (31).

We can state our result as follows: *If the elements y_i and z_i of a simultaneous fundamental system of* (1) *are made to undergo cogredient linear substitutions with constant coefficients and non-vanishing determinants, the corresponding solutions of the adjoined system of differential equations also undergo mutually cogredient linear substitutions with constant coefficients and with the same determinant. The coefficients of the second set of substitutions are the minors of those of the first set in their determinant, divided by the square root of that determinant. In a slightly modified sense then, the quantities (y_k, z_k) and (U_k, V_k) are contragredient. The quantities (y_k, z_k) and $(\lambda_k, \mu_k) = (u_k/D, v_k/D)$ are contragredient in the ordinary sense of the word.*

Upon this theorem rests the simple relation between the monodromic groups and the transformation groups of reciprocal systems of differential equations.

We can complete the relations (27) in an interesting manner. We have

$$\begin{aligned} y_k'' + p_{11} y_k' + p_{12} z_k' + q_{11} y_k + q_{12} z_k &= 0, \\ z_k'' + p_{21} y_k' + p_{22} z_k' + q_{21} y_k + q_{22} z_k &= 0 \end{aligned} \qquad (k = 1, 2, 3, 4).$$

If we multiply these equations by U_k and V_k successively and add, taking into account the relations (27) we shall find

$$\begin{aligned} -\sum_{k=1}^{4} y_k' U_k' = \sum_{k=1}^{4} y_k'' U_k &= +p_{12}\sqrt{D}, \\ \sum_{k=1}^{4} y_k'' V_k &= -p_{11}\sqrt{D}, \\ \sum_{k=1}^{4} z_k'' U_k &= +p_{22}\sqrt{D}, \\ -\sum_{k=1}^{4} z_k' V_k' = \sum_{k=1}^{4} z_k'' V_k &= -p_{21}\sqrt{D}. \end{aligned} \tag{34}$$

Also, if (27) and (34) are used,

$$(35)\qquad \sum_{k=1}^{4} y_k' V_k' = \frac{D'}{2\sqrt{D}} + p_{11}\sqrt{D}, \quad \sum_{k=1}^{4} z_k' U_k' = -\frac{D'}{2\sqrt{D}} - p_{22}\sqrt{D}.$$

Finally, using (21) as we have just used (1) to deduce (34), we have

$$(36)\qquad \begin{aligned} \sum_{k=1}^{4} U_k'' y_k &= +p_{12}\sqrt{D}, \quad \sum_{k=1}^{4} U_k'' z_k = -p_{11}\sqrt{D}, \\ \sum_{k=1}^{4} V_k'' y_k &= +p_{22}\sqrt{D}, \quad \sum_{k=1}^{4} V_k'' z_k = -p_{21}\sqrt{D}, \end{aligned}$$

as one sees also from the reciprocity of (1) and (21).

I have shown in former papers that every system of form (1) can be reduced to a *semi-canonical* form characterized by the conditions $p_{ik} = 0$ and also that the particular curves thus obtained on the integrating ruled surface are the curved asymptotic lines.

The above formulæ furnish a new proof for the latter theorem. As a matter of fact a simpler reduction is sufficient to determine the asymptotic lines. For, let the given system be transformed into another for which merely $p_{12} = p_{21} = 0$, while p_{11} and p_{22} may be arbitrary. Then the integral curves C_y and C_z on S will be such that

$$\sum_{k=1}^{4} U_k y_k = 0, \quad \sum_{k=1}^{4} U_k y_k' = 0, \quad \sum_{k=1}^{4} U_k y_k'' = 0,$$

$$\sum_{k=1}^{4} V_k z_k = 0, \quad \sum_{k=1}^{4} V_k z_k' = 0, \quad \sum_{k=1}^{4} V_k z_k'' = 0,$$

i. e., the plane tangent to S at (y_1, y_2, y_3, y_4) is the osculating plane of the curve C_y at that point, and the plane tangent to S at (z_1, z_2, z_3, z_4) is the osculating plane of C_z at that point. Therefore C_y and C_z are asymptotic lines of the surface.

If, then, in any system of form (1), $p_{12} = p_{21} = 0$, *its integral curves are asymptotic lines on its integrating ruled surface.*

If a given system (1) has by a first transformation been converted into another for which $p_{12} = p_{21} = 0$, the semi-canonical form, for which p_{11} and p_{22} also vanish, can be obtained very easily by putting

$$y = \eta e^{-\frac{1}{2}\int p_{11}\,dx}, \quad z = \zeta e^{-\frac{1}{2}\int p_{22}\,dx}.$$

Since such a transformation merely multiplies $y_1, \cdots, y_4$ by the same factor, and similarly $z_1, \cdots, z_4$, it does not affect the significance of these quantities as the homogeneous coördinates of corresponding points on two asymptotic lines.

It may be noted that, analytically, our theory rests upon the consideration of certain minors of the determinant D. By the consideration of other minors other similar theories may be developed. One of these theories has been already touched upon in a previous paper,* namely, that of the differential equation satisfied by the six determinants of the second order in the matrix

$$\begin{vmatrix} y_1 & y_2 & y_3 & y_4 \\ z_1 & z_2 & z_3 & z_4 \end{vmatrix},$$

i. e., by the six homogeneous line-coördinates of the generating straight line L_{yz} of the ruled surface S.

All of the systems of differential equations thus associated with (1) are of interest in connection with their geometrical interpretations, which are obvious enough. It is clear that a general theory of non-developable ruled surfaces can be conveniently built up as the geometric equivalent of the theory of a system of differential equations of form (1). This I have pointed out before, and a number of detached theorems of such a theory of ruled surfaces have already appeared in my papers. A systematic theory of ruled surfaces on such a basis appears to be a desideratum; and this I shall attempt to furnish on some future occasion.

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ON THE INVARIANTS OF QUADRATIC DIFFERENTIAL FORMS*

BY

CHARLES NELSON HASKINS

In the following paper I propose to investigate, by means of LIE's theory of continuous groups, the problem of determining the number of invariants of the general quadratic differential form in n variables.

§ 1. *Introduction.*

The n-ary quadratic differential forms

$$\phi \equiv \sum_{i=1}^{n} \sum_{k=1}^{i} a_{ik}(x_1 \cdots x_n)\, dx_i dx_k \qquad (a_{ik} \equiv a_{ki}), \tag{1}$$

the invariants of which are to be considered, will be subjected to the restriction that their discriminants, the determinants

$$a \equiv |a_{11} \cdots a_{nn}|, \tag{2}$$

shall not vanish identically. The coefficients a_{ik}, together with their partial derivatives of all orders, will be continuous functions of the n independent variables $x_1 \cdots x_n$.

These forms will be subjected to the infinite group of all point transformations in the variables $x_1 \cdots x_n$, which are generated by the infinitesimal transformation

$$Xf \equiv \sum_{r=1}^{n} \xi_r(x_1 \cdots x_n) \frac{\partial f}{\partial x_r}. \tag{3}$$

The quantities ξ_r, with their partial derivatives of all orders, will be supposed to be continuous functions of the n independent variables $x_1 \cdots x_n$, but, except for this restriction, wholly arbitrary. This infinite group of transformations includes all point transformations in the neighborhood of the identical transformation

$$x_i' = x_i \qquad (i = 1, 2, \cdots, n).$$

* Presented to the Society at the Ithaca meeting, August 20, 1901. Received for publication June 17, 1901.

Under a transformation

$$x_i' = x_i'(x_1 \cdots x_n) \qquad (i=1, 2, \cdots, n), \tag{4}$$

the form ϕ becomes

$$\phi' \equiv \sum_{i=1}^{n}\sum_{k=1}^{n} a'_{ik}(x_1' \cdots x_n')\,dx_i'\,dx_k' \qquad (a'_{ki} \equiv a'_{ik}), \tag{5}$$

where

$$a'_{ik} = \sum_{r=1}^{n}\sum_{s=1}^{n} a_{rs}\frac{\partial x_r}{\partial x_i'}\frac{\partial x_s}{\partial x_k'} \qquad (i, k=1, 2, \cdots, n). \tag{6}$$

An *invariant* of the form ϕ under the above group is a function f of the variables x, of the coefficients a_{ik} and of their derivatives of various orders, taken with respect to the variables x, such that, whether it be constructed from the original variables x and the coefficients and derivatives from the original form ϕ, or from the variables x' and the coefficients and derivatives from the transformed form ϕ', its value is the same whatever transformation of the group is employed. (It will appear in the course of the work that the variables x cannot enter explicitly.)

The *order* of an invariant is the order of the highest derivative appearing in it.

The problem of the present paper is that of the determination of the number of invariants of each order μ for the general form in n variables.

It will be shown that such a form has

$$J_{n2} = \frac{(n-2)(n-1)n(n+3)}{12} \tag{7}$$

invariants of order two, and

$$I_{n\mu} = n\frac{\mu-1}{2}\frac{(n+\mu-1)!}{(n-2)!\,(\mu+1)!} \tag{8}$$

invariants of order $\mu > 2$, provided $n \geqq 3$. The latter formula holds also when $\mu > 3$, if $n = 2$.

The cases $\mu = 0$, $\mu = 1$ have been treated by RICCI,* who showed that no invariants of these orders exist.

The case $n = 2$ has been fully studied by ŻORAWSKI,† who showed that there is in this case one invariant of order two, one of order three, and $\mu - 1$ of order $\mu > 3$.

LEVI-CIVITA‡ has found a lower limit for the number of invariants of a given order for the general form. He expressly states,§ however, that he does

* RICCI, Annali di Matematica, ser. 2, vol. 14 (1886), pp. 4–5.

† ŻORAWSKI, Acta Mathematica, vol. 16 (1892), p. 41.

‡ LEVI-CIVITA, Reale Istituto Veneto, Atti, ser. 7, vol. 5_2 (1894), pp. 1468, 1507.

§ Ibid., p. 1507.

not determine the exact number. It is to this determination that the present paper is devoted.

ŻORAWSKI and LEVI-CIVITA have used in their work the methods of LIE. By these methods the invariants of the form are all to be found as solutions of complete systems of homogeneous linear partial differential equations of the first order. Hence the number of independent invariants is equal to the number of independent solutions of the complete systems, and this, in turn, is equal to the excess of the number of variables in the equations over the number of independent equations. Hence the entire determination of the number of independent invariants of the form reduces to the determination of the number of independent equations in the complete system which the invariants must satisfy. It is precisely this determination which is omitted from the work of LEVI-CIVITA.

The direct method of finding the number of these independent equations is the computation of determinants from the matrix of the system. The great number of the equations and their somewhat complicated form renders this method, however, impracticable. Hence it is necessary to have recourse to a special method particularly adapted to the type of equations under consideration. This method is, in outline, the following. By means of a simple lemma the problem is reduced to the determination of the independence of the equations of two different sets. Each of these determinations can be carried on by methods of mathematical induction. By suitable changes of the variables and by the use of certain particular forms ϕ the work is materially simplified.

§ 2. *Differential equations of the problem.*

In order to determine by LIE's methods the invariants of order μ we must first "extend" the group Xf to the coefficients a_{ik} of the form and to their derivatives of order μ and lower.* To do this we note that, under the extended transformation, ϕ does not change its value, and consequently, if we denote by $X_0 f$ the transformation Xf extended to the coefficients a_{ik}, we have

$$(1) \qquad X_0\phi \equiv \sum_{i=1}^{n}\sum_{k=1}^{n}\left\{ Xa_{ik} + \sum_{r=1}^{n}\left(a_{kr}\frac{\partial \xi_r}{\partial x_i} + a_{ir}\frac{\partial \xi_r}{\partial x_k}\right)\right\} dx_i dx_k \equiv 0,$$

the identity having reference to the n differentials $dx_1, \cdots, dx_n$.

Hence we have†

$$(2) \qquad Xa_{ik} + \sum_{r=1}^{n}\left(a_{kr}\frac{\partial \xi_r}{\partial x_i} + a_{ir}\frac{\partial \xi_n}{\partial x_k}\right) = 0.$$

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*Cf. LIE, Mathematische Annalen, vol. 24 (1884), p. 575. ŻORAWSKI, Acta Mathematica, vol. 16 (1892), p. 9.

†Cf. KILLING, Crelle's Journal, vol. 109 (1891), p. 121.

From this the extension to the derivatives is obtained by successive applications of the formula

$$X\left(\frac{\partial^{m+1}a_{ik}}{\partial x_{l_1}\cdots\partial x_{l_m}\partial x_{l_{m+1}}}\right)=\frac{\partial}{\partial x_{l_{m+1}}}\left\{X\left(\frac{\partial^m a_{ik}}{\partial x_{l_1}\cdots\partial x_{l_m}}\right)\right\} \tag{3}$$
$$-\sum_{r=1}^{n}\frac{\partial}{\partial x_r}\left(\frac{\partial^m a_{ik}}{\partial x_{l_1}\cdots\partial x_{l_m}}\right)\frac{\partial \xi_r}{\partial x_{l_{m+1}}}.*$$

We shall use the following notations:

$$\underset{l_1\cdots l_\mu}{a_{ik}}\equiv\frac{\partial^\mu a_{ik}}{\partial x_{l_1}\cdots\partial x_{l_\mu}},\qquad \underset{l_1\cdots l_\mu}{\mathfrak{a}_{ik}}\equiv X\,\underset{l_1\cdots l_\mu}{a_{ik}},$$
$$\underset{l_1\cdots l_\mu}{p_{ik}}\equiv\frac{\partial f}{\partial\left(\underset{l_1\cdots l_\mu}{a_{ik}}\right)},\qquad \underset{l_1\cdots l_\mu}{\xi_r}\equiv\frac{\partial^\mu \xi_r}{\partial x_{l_1}\cdots\partial x_{l_\mu}}, \tag{4}$$

and shall say that each of these quantities is of order μ. For convenient use in the text, however, the notation (4) will sometimes be replaced respectively by the notations:

$$a_{ik\,l_1\cdots l_\mu},\quad \mathfrak{a}_{ik\,l_1\cdots l_\mu},\quad p_{ik\,l_1\cdots l_\mu},\quad \xi_{r\,l_1\cdots l_\mu}. \tag{4'}$$

These notations differ somewhat from those used by LIE and ŻORAWSKI. They are, however, essentially the same as those of LEVI-CIVITA and have the advantage that they make obvious certain relations which hold when for the derivatives of the coefficients a_{ik} are substituted the well-known symbols of CHRISTOFFEL.

The invariants of order μ or lower are all solutions of the linear partial differential equations obtained by equating to zero the coefficients of the quantities ξ and their derivatives in the expression

$$\overline{Xf}\equiv Xf+\sum_{i=1}^{n}\sum_{k=1}^{i}\mathfrak{a}_{ik}p_{ik}+\sum_{i=1}^{n}\sum_{k=1}^{i}\sum_{l_1=1}^{n}\underset{l_1}{\mathfrak{a}_{ik}}\underset{l_1}{p_{ik}}+\cdots \tag{5}$$
$$+\sum_{i=1}^{n}\sum_{k=1}^{i}\sum_{l_1=1}^{n}\sum_{l_2=1}^{l_1}\cdots\sum_{l_\mu=1}^{l_{\mu-1}}\underset{l_1\cdots l_\mu}{\mathfrak{a}_{ik}}\underset{l_1\cdots l_\mu}{p_{ik}}.$$

These equations form a complete system. † The coefficients of the quantities ξ give

$$\frac{\partial f}{\partial x_1}=\frac{\partial f}{\partial x_2}=\cdots=\frac{\partial f}{\partial x_n}=0. \tag{6}$$

That is, the invariants cannot contain the variables x explicitly.

* LIE-ENGEL, *Theorie der Transformationsgruppen*, vol. 1, p. 545.

† LIE, Mathematische Annalen, vol. 24 (1884), p. 566.

It is easily seen that the system remains complete after these equations are stricken out. We have, therefore, to discuss the complete system obtained by equating to zero the coefficients of the ξ's and of their derivatives in the expression

$$X_\mu f \equiv \sum_{i=1}^{n}\sum_{k=1}^{i} a_{ik}\, p_{ik} + \sum_{i=1}^{n}\sum_{k=1}^{i}\sum_{l_1=1}^{n} a_{ik\,l_1}\, p_{ik\,l_1} + \cdots + \sum_{i=1}^{n}\sum_{k=1}^{i}\sum_{l_1=1}^{n}\sum_{l_2=1}^{l_1}\cdots\sum_{l_\mu=1}^{l_{\mu-1}} a_{ik\,l_1\cdots l_\mu}\, p_{ik\,l_1\cdots l_\mu}. \tag{7}$$

We shall call the equations of this system 'the equations of order μ.'

We find

$$a_{ik} = -\sum_{r=1}^{n}\left(a_{ir}\,\xi_{r\,k} + a_{kr}\,\xi_{r\,i}\right), \tag{8}$$

$$a_{ik\,l_1} = -\sum_{r=1}^{n}\left\{\left(a_{ir\,l_1}\,\xi_{r\,k} + a_{kr\,l_1}\,\xi_{r\,i} + a_{ik\,r}\,\xi_{r\,l_1}\right) + \left(a_{ir}\,\xi_{r\,kl_1} + a_{kr}\,\xi_{r\,il_1}\right)\right\}, \tag{9}$$

$$\begin{aligned} a_{ik\,l_1l_2} = -\sum_{r=1}^{n}\Big\{&\left(a_{ir\,l_1l_2}\,\xi_{r\,k} + a_{kr\,l_1l_2}\,\xi_{r\,i} + a_{ik\,rl_2}\,\xi_{r\,l_1} + a_{ik\,rl_1}\,\xi_{r\,l_2}\right)\\ &+ \left(a_{ir\,l_2}\,\xi_{r\,kl_1} + a_{ir\,l_1}\,\xi_{r\,kl_2} + a_{kr\,l_2}\,\xi_{r\,il_1} + a_{kr\,l_1}\,\xi_{r\,il_2} + a_{ik\,rl_1}\,\xi_{r\,l_2}\right)\\ &+ \left(a_{ir}\,\xi_{r\,kl_1l_2} + a_{kr}\,\xi_{r\,il_1l_2}\right)\Big\}, \end{aligned} \tag{10}$$

etc.

The formulæ giving the values of the a's of higher orders are important only so far as the terms containing the highest derivatives of the ξ's are concerned. These are easily seen to be, for $a_{ik\,l\cdots l_m}$,

$$-\sum_{r=1}^{n}\left(a_{ir}\,\xi_{r\,kl_1\cdots l_m} + a_{kr}\,\xi_{r\,il_1\cdots l_m}\right). \tag{11}$$

The above expressions for $X_\mu f$ and the a's make evident that peculiarity of the equations determining the invariant which renders possible the use of the special method of the present paper in determining their independence. For it is easily seen that, since the a's of order $\mu+1$ contain derivatives of the ξ's of all orders $1, 2, \cdots, \mu+2$, while the a's of order μ contain derivatives of the ξ's of orders $1, 2, \cdots, \mu+1$ only, the equations of order $\mu+1$ may be obtained by annexing certain terms to the equations of order μ, and by adding certain equations to the system. The annexed terms and the added equations come from the expression

$$X_{\mu+1}f - X_\mu f \equiv \sum_{i=1}^{n}\sum_{k=1}^{i}\sum_{l_1=1}^{n}\sum_{l_2=1}^{l_1}\cdots\sum_{l_\mu=1}^{l_{\mu-1}}\sum_{l_{\mu+1}=1}^{l_\mu} a_{ik\,l_1\ldots l_\mu l_{\mu+1}}\, p_{ik\,l_1\ldots l_\mu l_{\mu+1}}, \tag{12}$$

and hence contain no p's of order less than $\mu + 1$, *i. e.*, they contain none of the p's which appear in the equations of order μ or lower.

The added equations we shall call the *final* equations of order μ.

The equations of order μ which are obtained by annexing terms to the equations of order zero we call the *primary* equations of order μ. The remaining equations of order μ we call the *secondary* equations of that order.

§ 3. *Decomposition of the problem.*

The form of the equations under discussion permits us to decompose the problem of determining their independence into two auxiliary problems. The possibility of this decomposition is a consequence of the following

LEMMA.—*If all the equations of order $\mu - 1$ are independent, and if the final equations of order μ are all independent, then all the equations of order μ are independent.*

For suppose there are M equations of order $\mu - 1$ and N final equations of order μ. Then there are $M + N$ equations of order μ. At least one determinant, Δ_1, of order M, formed from the matrix of the equations of order $\mu - 1$, does not vanish; and at least one determinant, Δ_2, of order N, formed from the matrix of the final equations of order μ does not vanish. Form now that determinant, Δ_3, of order $M + N$, the first M columns of which contain in their first M rows the elements of Δ_1, and the last N columns of which contain in their last N rows the elements of Δ_2. Then, since the final equations of order μ contain no p's of order lower than μ, the elements in the first M columns and last N rows of Δ_3 are all zero. Hence

$$\Delta_3 = \Delta_1 \Delta_2 \neq 0, \tag{1}$$

and consequently, since at least one determinant of order $M + N$ formed from the matrix of the equations of order $\mu + 1$ does not vanish, these equations are all independent.

This theorem was used by ŻORAWSKI* for the simple case of the binary forms. It allows us to resolve the problem under discussion into two parts, viz.:

I. *The determination of the independence of the final equations of general order μ.*

II. *The determination of the existence of an order μ for which all the equations are independent.*

It will appear that for $n \geqq 3$ the equations of order two are all independent. For $n = 2$ this is not true, but the equations of order three are all independent.

* ŻORAWSKI, Acta Mathematica, vol. 16 (1892), p. 21.

§ 4. *Independence of the final equations.*

We shall first show that if the final equations of order $\mu+1$ are not independent the same is true of the final equations of order μ.

The final equations of order $\mu+1$ are readily seen to be obtained by equating to zero the coefficient of the $(\mu+2)$th derivatives of the ξ's in the expression

$$(1)\qquad \Gamma_{\mu+1}f=\sum_{i,k}\sum_{l_1,l_2,\cdots,l_{\mu+1}} p_{ik\,l_1l_2\cdots l_\mu l_{\mu+1}}\sum_r\left(a_{ir}\,\xi_{r\,kl_1\cdots l_\mu l_{\mu+1}}+a_{kr}\,\xi_{r\,il_1\cdots l_\mu l_{\mu+1}}\right)$$
$$(i,k,l_1,\cdots,l_{\mu+1}=1,2,\cdots,n;\ k\leqq i,\ l_1\geqq l_2\geqq\cdots\geqq l_\mu\geqq l_{\mu+1}).$$

If these equations are dependent there exists a set of multipliers, not all zero, such that, if they be applied to the corresponding equations and the results added, the coefficient of each p is identically zero. But since, in forming the equations, we collect the coefficients of the various quantities $\xi_{r\,il_1\cdots l_\mu l_{\mu+1}}$, and equate them to zero, it follows that if we consider these quantities, not as derivatives of the n ξ's, but as undetermined multipliers, then the necessary and sufficient condition that the final equations of order $\mu+1$ shall not be all independent is that there shall exist a set of quantities $\xi_{r\,il_1\cdots l_\mu l_{\mu+1}}$, not all zero, such that they make the expression $\Gamma_{\mu+1}f$, considered as a polynomial in the variables $p_{ik\,l_1l_2\cdots l_\mu l_{\mu+1}}$, vanish identically. That is, there must exist a set of quantities $\xi_{r\,il_1\cdots l_\mu l_{\mu+1}}$, not all identically zero, satisfying the equations

$$(2)\qquad \sum_{r=1}^{n}\left(a_{ir}\,\xi_{r\,kl_1\cdots l_\mu l_{\mu+1}}+a_{kr}\,\xi_{r\,il_1\cdots l_\mu l_{\mu+1}}\right)=0$$
$$(i,k,l_1,\cdots,l_{\mu+1}=1,2,\cdots,n;\ k\leqq i,\ l_1\geqq l_2\geqq\cdots\geqq l_\mu\geqq l_{\mu+1}).$$

Suppose that such a set of multipliers exists. There is, therefore, at least one index λ, appearing as the last index $l_{\mu+1}$ for which the corresponding equations are satisfied by multipliers $\xi_{r\,il_1\cdots l_\mu l_{\mu+1}}$, not all zero. The ξ's appearing in the other equations may or may not vanish.

Now it amounts merely to a renaming of the x's if we take x_λ as x_1. Hence there is a set of equations:

$$(3)\qquad \sum_{r=1}^{n}\left(a_{ir}\,\xi_{r\,kl_1\cdots l_\mu 1}+a_{kr}\,\xi_{r\,il_1\cdots l_\mu 1}\right)=0$$
$$(i,k,l_1,\cdots,l_\mu=1,2,\cdots,n;\ k\leqq i,\ l_1\geqq l_2\geqq\cdots\geqq l_\mu),$$

satisfied by a set of quantities $\xi_{r\,il_1\cdots l_{\mu 1}}$, not all of which are zero. But then, putting

$$(4)\qquad \eta_{r\,il_1\cdots l_\mu}=\xi_{r\,il_1\cdots l_\mu 1},$$

we have

$$(5)\qquad \sum_{r=1}^{n}\left(a_{ir}\,\eta_{r\,kl_1\cdots l_\mu}+a_{kr}\,\eta_{r\,il_1\cdots l_\mu}\right)=0$$
$$(i,k,l_1,\cdots,l_\mu=1,2,\cdots,n;\ k\leqq i,\ l_1\geqq l_2\geqq\cdots\geqq l_\mu).$$

These are, however, precisely the equations which, if satisfied by a set of η's not all zero, give the necessary and sufficient condition that the final equations of order μ shall not be all independent. Hence we have the following

THEOREM: *If the final equations of order $\mu+1$ are not all independent, the final equations of order μ are not all independent.*

COROLLARY: *If the final equations of order $\mu>1$ are not all independent, the final equations of order unity are not all independent.*

It may now be shown by means of a device already used for a similar purpose by RICCI,* though in connection with finite rather than infinitesimal transformations, that the final equations of order unity are all independent. This device consists in substituting for the derivatives, $a_{ik\,|\,l}$, of the coefficients of the form, the well-known three-index symbols

$$\left[\begin{matrix} ik \\ l \end{matrix}\right] \equiv \tfrac{1}{2}\left(a_{\underset{k}{il}} + a_{\underset{i}{kl}} - a_{\underset{l}{ik}}\right) \tag{6}$$

of CHRISTOFFEL. The equations obtained from the final equations of order one by means of this change of variables are easily shown to be independent, and hence the original equations must have been so.

The three-index symbols satisfy the relations

$$\left[\begin{matrix} ik \\ l \end{matrix}\right] = \left[\begin{matrix} ki \\ l \end{matrix}\right], \quad a_{\underset{l}{ik}} = \left[\begin{matrix} il \\ k \end{matrix}\right] + \left[\begin{matrix} kl \\ i \end{matrix}\right]. \tag{7}$$

Hence there are $n^2(n+1)/2$ distinct symbols, and they can replace the system of the $n^2(n+1)/2$ first derivatives, $a_{ik\,|\,l}$, of the coefficients of the form.

Putting

$$q_{\underset{l}{ik}} \equiv \frac{\partial f}{\partial \left[\begin{matrix} ik \\ l \end{matrix}\right]}, \tag{8}$$

we find that

$$\Gamma_1 f \equiv \sum_{i=1}^{n}\sum_{k=1}^{i}\sum_{l=1}^{n} p_{\underset{l}{ik}} \sum_{r=1}^{n} (a_{\underset{k}{ir}}\xi_r + a_{\underset{i}{kr}}\xi_r) \tag{9}$$

becomes

$$\Gamma_1 f \equiv \sum_{i=1}^{n}\sum_{k=1}^{i}\sum_{l=1}^{n} q_{\underset{l}{ik}} \sum_{r=1}^{n} a_{lr}\xi_{\underset{ik}{r}} \equiv \sum_{i=1}^{n}\sum_{k=1}^{i}\sum_{r=1}^{n} \xi_{\underset{ik}{r}} \sum_{l=1}^{n} a_{lr} q_{\underset{ik}{l}}, \tag{10}$$

and hence the final equations of order unity are transformed into

$$\sum_{l=1}^{n} a_{lr} q_{\underset{l}{ik}} = 0 \qquad (i, k, r = 1, 2, \cdots, n;\ k \leqq i). \tag{11}$$

* RICCI, Annali di Matematica, ser. 2, vol. 14 (1886), p. 5.

These $n^2(n+1)/2$ equations contain $n^2(n+1)/2$ variables $q_{ik|l}$ and hence, as their determinant is

$$a^{\frac{n(n+1)}{2}} \not\equiv 0, \tag{12}$$

they are all independent. We have, then, the following

THEOREMS.—(*a*) *The final equations of order unity are all independent.* Hence (*b*) *The final equations of all orders* $\mu \geqq 1$ *are all independent.* Hence (*c*) *The secondary equations of all orders* $\mu \geqq 1$ *are all independent.*

It may be noted in passing that the proofs in this section have not depended on any properties of the derivatives of the coefficients of the form nor on the coefficients themselves, except in that the discriminant a of the form shall not vanish.

§ 5. *Determination of an order* μ *for which all equations are independent.*

We shall now show that if $n \geqq 3$ all the equations of order $\mu = 2$ are independent. We shall also obtain the result of ŻORAWSKI, that for $n = 2$ the equations of order $\mu = 2$ are not independent, but that the equations of order $\mu = 3$ are independent. It will be found possible, however, to simplify considerably ŻORAWSKI'S proof.

In this part of the work we can only show that the equations are independent for the general form. They may not be so for special forms. As we are seeking, however, the number of functions which are invariant when constructed from any form whatever, the dependence of the equations in the case of special forms is immaterial.

The method of proof is, as in the foregoing section, that of mathematical induction. If the equations of order two are not independent, then, whatever form the form ϕ may be, there can be found multipliers not all zero, such that when they are applied to the corresponding equations and the results added, they reduce to zero the coefficients of the variables p. We find the equations which these multipliers must satisfy. Then, if for any particular form ϕ there exist no multipliers, not identically zero, satisfying these equations, the independence is established. For the hypothesis was that such multipliers, not all zero, exist for every form. By the use of a certain particular type of form we are able to show that if the equations are not independent for the form in n variables they are not independent for the form in $n-1$ variables. The reasoning may be applied again and again till the form in three variables is reached. We then show directly that the equations for the form in three variables are all independent. Hence it follows that the equations for the form in $n > 3$ variables are also independent.

As in the foregoing section we find that the introduction of new variables simplifies the work. We introduce the four-index symbols of CHRISTOFFEL,

$$(\lambda\mu, \nu\rho) \equiv \tfrac{1}{2}\left(a_{\substack{\lambda\rho \\ \mu\nu}} + a_{\substack{\mu\nu \\ \lambda\rho}} - a_{\substack{\lambda\nu \\ \mu\rho}} - a_{\substack{\mu\rho \\ \lambda\nu}}\right)$$

$$+ \sum_{l=1}^{n}\sum_{m=1}^{n} A_{lm}\left(\begin{bmatrix}\lambda\rho \\ l\end{bmatrix}\begin{bmatrix}\mu\nu \\ m\end{bmatrix} - \begin{bmatrix}\lambda\nu \\ l\end{bmatrix}\begin{bmatrix}\mu\rho \\ m\end{bmatrix}\right), \tag{1}$$

and find that we have thereby partly solved the problem of determining the form of the invariants. For *

$$\beta_{\lambda\mu,\,\nu\rho} \equiv X(\lambda\mu, \nu\rho)$$

$$= -\sum_{r=1}^{n}\left\{(r\mu, \nu\rho)\,\xi_{\substack{r \\ \lambda}} + (\lambda r, \nu\rho)\,\xi_{\substack{r \\ \mu}} + (\lambda\mu, r\rho)\,\xi_{\substack{r \\ \nu}} + (\lambda\mu, \nu r)\,\xi_{\substack{r \\ \rho}}\right\}, \tag{2}$$

and this expression does not contain the second and third derivatives of the ξ's. Hence the four-index symbols all satisfy the secondary equations of order two. But there are $N = n^2(n^2-1)/12$ independent four-index symbols,† and the secondary equations of order two can have at most N indeperdent common solutions. For the number N_{n2} of variables in these equations, all of which are independent, is equal to the number of the first and second derivatives of the coefficients, a_{ik}, of the form; and the number M_{n2} of equations is equal to the number of second and third derivatives of the quantities ξ. The difference, $N_{n2} - M_{n2}$, gives the maximum number of independent common solutions of the equations. But

$$N_{n2} = \frac{n^2(n+1)(n+3)}{4}, \quad M_{n2} = \frac{n^2(n+1)(n+5)}{6}, \tag{3}$$

therefore

$$N_{n2} - M_{n2} = \frac{n^2(n^2-1)}{12} = N. \tag{4}$$

Hence the invariants of order two can contain the first and second derivatives of the coefficients a_{ik} only through the N independent four-index symbols. Hence by the introduction of these symbols as new variables, the system of equations determining the invariants of order two is reduced to the system of n^2 equations obtained by introducing the N independent four-index symbols into the n^2 primary equations of order two.

These equations form a complete system. If relations exist among them an equal number of relations exist among the original equations. The problem, then, is reduced to that of determining the independence of these n^2 equations.

* Cf. LEVI-CIVITA, R. Ist. Veneto, Atti, ser. 7, vol. 5_2 (1894), p. 1454.

† CHRISTOFFEL, Crelle's Journal, vol. 70 (1869), p. 54.

The equations are obtained by equating to zero the coefficients of the n^2 quantities $\xi_{r\,|\,s}$ in the expression

$$(5) \qquad Yf \equiv \sum_{i=1}^{n}\sum_{k=1}^{i} a_{ik}\frac{\partial f}{\partial a_{ik}} + \sum_{\lambda=1}^{n}{}'\sum_{\mu=1}^{\lambda-1}{}'\sum_{\nu=1}^{n}{}'\sum_{\rho=1}^{\nu-1}{}' \beta_{\lambda\mu\,,\,\nu\rho}\frac{\partial f}{\partial(\lambda\mu\,,\,\nu\rho)}.$$

Here the accents on the signs $\sum$ indicate that the summation extends only to those values of λ, μ, ν, ρ giving the independent symbols.

Introducing the values of a_{ik}, $\beta_{\lambda\mu\,,\,\nu\rho}$ already found we have

$$\begin{aligned}(6) \qquad -Yf \equiv{}& \sum_{i=1}^{n}\sum_{k=1}^{i}\frac{\partial f}{\partial a_{ik}}\sum_{r=1}^{n}\left(a_{ir}\underset{k}{\xi}_r + a_{kr}\underset{i}{\xi}_r\right)\\ &+ \sum_{\lambda=1}^{n}{}'\sum_{\mu=1}^{\lambda-1}{}'\sum_{\nu=1}^{n}{}'\sum_{\rho=1}^{\nu-1}{}'\frac{\partial f}{\partial(\lambda\mu,\,\nu\rho)}\sum_{r=1}^{n}\Big\{(r\mu,\,\nu\rho)\underset{\lambda}{\xi}_r + (\lambda r,\,\nu\rho)\underset{\mu}{\xi}_r\\ &\qquad + (\lambda\mu,\,r\rho)\underset{\mu}{\xi}_r + (\lambda\mu,\,\nu r)\underset{\rho}{\xi}_r\Big\},\end{aligned}$$

and we find that, as in the case of the final equations, the condition that the equations shall be dependent is the existence of a set of multipliers $\xi_{r\,|\,s}$, not all zero, satisfying the equations

$$(A) \qquad -a_{ik} \equiv \sum_{r=1}^{n}\{a_{ir}\underset{k}{\xi}_r + a_{kr}\underset{i}{\xi}_r\} = 0 \qquad (i,\,k=1,\,2,\,\cdots,\,n\,;\,k\leqq i),$$

$$(B) \qquad -\beta_{\lambda\mu\,,\,\nu\rho} = \sum_{r=1}^{n}(r\mu,\,\nu\rho)\underset{\lambda}{\xi}_r + (\lambda r,\,\nu\rho)\underset{\mu}{\xi}_r + (\lambda\mu,\,r\rho)\underset{\nu}{\xi}_r + (\lambda\mu,\,\nu r)\underset{\rho}{\xi}_r\} = 0$$
$$(\lambda,\,\mu,\,\nu,\,\rho=1,\,2,\,\cdots,\,n\,;\,\mu<\lambda,\,\rho<\nu).$$

The quantities $\xi_{r\,|\,s}$, if they exist, must be such functions of the quantities a_{ik} and of the four-index symbols that, if substituted in (A) and (B) they satisfy these equations identically. Thus if, for any single form, arbitrarily chosen save for the continuity of the coefficients and the non-vanishing of the discriminant, it happens that we find that there exist no n^2 quantities $\xi_{r\,|\,s}$, not all identically zero, satisfying these equations, then the equations determining the invariants of order two are all independent.

Consider, then, the special form,

$$\phi = \phi_1 + dx_n^2 \qquad (a=1\cdot a_1 \neq 0),$$

where

$$\phi_1 = \sum_{i=1}^{n-1}\sum_{k=1}^{n-1} a_{ik}(x_1\cdots x_{n-1})\,dx_i dx_k \qquad (a_1 = |\,a_{11}\cdots a_{n-1\,n-1}\,|\,),$$

Here ϕ_1 is a form in the $n-1$ variables $x_1, \cdots, x_{n-1}$, only. It is easily seen that for such a special form ϕ,

$$\begin{bmatrix} rn \\ s \end{bmatrix} \equiv 0, \ (nh, ik) \equiv 0 \qquad (r, h, i, k = 1, 2, \cdots, n), \tag{7}$$

$$a_{rn} = 0 \qquad (r \neq n), \tag{8}$$

$$a_{nn} = 1.$$

Making use of these relations we find that equations (A) and (B) can be divided into sets as follows:

I. $$-a_{ik} \equiv \sum_{r=1}^{n-1} \left\{ a_{rk} \underset{i}{\xi_r} + a_{ri} \underset{k}{\xi_r} \right\} = 0 \qquad (i, k = 1, 2, \cdots, n-1; k \leqq i).$$

II. $$-\beta_{\lambda\mu, \nu\rho} \equiv \sum_{r=1}^{n-1} \left\{ (r\mu, \nu\rho) \underset{\lambda}{\xi_r} + (\lambda r, \nu\rho) \underset{\mu}{\xi_r} + (\lambda\mu, r\rho) \underset{\nu}{\xi_r} + (\lambda\mu, \nu r) \underset{\rho}{\xi_r} \right\}$$
$$(\lambda, \mu, \nu, \rho = 1, 2, \cdots, n-1; \mu > \lambda, \rho < \nu).$$

III. $$-a_{nk} \equiv \sum_{r=1}^{n-1} \left\{ a_{rk} \underset{n}{\xi_r} + a_{nn} \underset{k}{\xi_n} \right\} = 0 \qquad (k = 1, 2, \cdots, n-1; a_{nn} = 1).$$

IV. $$-a_{nn} \equiv 2a_{nn} \underset{n}{\xi_n} = 0 \qquad (a_{nn} = 1 \neq 0).$$

V. $$-\beta_{\lambda\mu, \nu\rho} \equiv \sum_{r=1}^{n-1} (\lambda\mu, \nu r) \underset{n}{\xi_n} = 0 \qquad (\lambda, \mu, \nu \leqq n-1).$$

VI. $$-\beta_{\lambda\mu, \nu\rho} \equiv 0 \qquad (\text{At least two indices, } \lambda, \mu, \nu, \rho \text{ are equal to } n).$$

We note that equations I, II are those which determine the multipliers for the equations satisfied by the invariants of order two for a form in $n-1$ variables.

Let us assume now that the equations which determine the invariants of order two for the general form in n variables are not all independent. Then they are, for the particular form ϕ, not all independent, whatever form in $n-1$ variables with non-vanishing discriminant the form ϕ_1 may be.

For such a form ϕ there exist, then, n^2 quantities, $\xi_{r\,|\,s}$, not all zero, satisfying equations I $\cdots$ VI.

Suppose now that the $(n-1)^2$ quantities $\xi_{r\,|\,s} (r, s \equiv n-1)$ are all identically zero. By IV, $\xi_{n\,|\,n} \equiv 0$. Hence not all the $2(n-1)$ quantities

$$\xi_{1\,|\,n}, \cdots, \xi_{n-1,\,1\,|\,n}, \xi_{n_1\,|\,1}, \cdots, \xi_{n\,|\,n-1},$$

are identically zero. It follows from III that, since $a_{nn} = 1 \neq 0$, and the determinant of the quantities $\xi_{r\,|\,n}$ is $a_1 \neq 0$, the quantities $\xi_{1\,|\,n}, \cdots, \xi_{n-1\,|\,n}$ all vanish if the quantities $\xi_{n\,|\,1}, \cdots, \xi_{n\,|\,n-1}$ all vanish, and conversely.

Hence for the forms under consideration not all the $n-1$ quantities $\xi_{r\,n}$ $(r=1, 2, \cdots, n-1)$ are identically zero. But they must satisfy equations V, and in particular, the following $n-1$ of those equations:

$$-\beta_{1\mu,\,1n} \equiv \underset{n}{\xi_1} + \sum_{r=2}^{n-1} (1\mu, 1r)\,\underset{n}{\xi_r} = 0 \qquad (\mu = 2, 3, \cdots, n-1), \tag{9}$$

$$-\beta_{23,\,2n} \equiv (23, 21)\,\underset{n}{\xi_1} + 0\,\underset{n}{\xi_2} + \sum_{r=3}^{n-1} (23, 2r)\,\underset{n}{\xi_r} = 0. \tag{10}$$

Let R be the determinant of this system of equations. Then, since by hypothesis not all the quantities $\xi_{r\,n}$ are identically zero, we must have

$$R \equiv 0. \tag{11}$$

But

$$R \equiv (23, 21) \begin{vmatrix} (12, 12), & (12, 13), & \cdots, & (12, 1n-1) \\ (13, 12), & (13, 13), & \cdots, & (13, 1n-1) \\ \cdot & \cdot & \cdots & \cdot \\ (1n-1, 12), & (1n-1, 13), & \cdots, & (1n-1, 1n-1) \end{vmatrix}. \tag{12}$$

Now referring to the definition of the four-index symbols we easily see that we may take the form of ϕ_1 so that, at a given point,

$$\begin{gathered} (1\mu, 1r) = (1r, 1\mu) = 0 \qquad (r \neq \mu), \\ (1\mu, 1\mu) = 1, \quad (23, 21) = 1. \end{gathered} \tag{13}$$

Then we have, at that point,

$$R = 1. \tag{14}$$

Hence

$$R \neq 0, \tag{15}$$

and we therefore have a contradiction.

It therefore follows that for all forms for which $R \neq 0$ all the $n-1$ quantities $\xi_{r\,n}(r=1, 2, \cdots, n-1)$ vanish identically and hence, by III and IV, all the n quantities $\xi_{n\,r}(r=1, 2, \cdots, n)$ must also vanish identically. But for all these forms the equations of order two are not all independent. Hence not all the $(n-1)^2$ functions $\xi_{r\,s}(r, s \leqq n-1)$ can be identically zero. But these quantities satisfy equations I and II; and we have therefore the result:

If the equations determining the invariants of order two are, for all forms in n variables, not independent, then there exist, for all forms in $n-1$ variables which satisfy the relations

$$R \neq 0, \qquad a_1 \neq 0, \tag{16}$$

$(n-1)^2$ quantities $\xi_{rs}(r, s \leqq n-1)$, not all identically zero, satisfying equations I and II.

Now the condition that equations I and II shall admit a set of solutions ξ_{rs}, not all identically zero, is the vanishing of a certain number, p, of determinants formed from the matrix of their coefficients. The elements of these determinants are either coefficients $a_{ik}(i, k = 1, 2, \cdots, n-1)$ of the form ϕ, or four-index symbols $(\lambda\mu, \nu\rho)$, $(\lambda, \mu, \nu, \rho \leqq n-1)$. Moreover the condition is that all the p determinants vanish. They vanish for all values of the arguments a_{ik}, $(\lambda\mu, \nu\rho)$, which make

$$R \neq 0, \qquad a_1 \neq 0, \tag{17}$$

and they are polynomials in those arguments. Let η be the number of arguments; ϵ, the highest degree to which any argument appears. Then we can certainly find, in the neighborhood of these values of the arguments which make

$$a_1 \neq 0, \qquad R = 0, \tag{18}$$

$\epsilon + 1$ values of each of the η arguments such that, for any combination of these values,

$$a_1 \neq 0, \qquad R \neq 0. \tag{19}$$

But for each of these combinations of the arguments there exist mutipliers ξ_{rs}, satisfying I and II and not all zero. Hence the p determinants in question vanish. But then, by a well-known theorem concerning polynomials, they vanish identically and hence, in particular, they vanish when $R = 0$. Hence the system of equations I and II admits a set of solutions, not all of which are zero, whether the form makes $R = 0$ or not. Hence the equations for the invariants of the second order for the form ϕ_1 are not all independent.

We have, therefore, the following

THEOREM.—*If the equations of order two are dependent for every form in n variables, they are dependent for every form in $n-1$ variables.*

COROLLARY.—*If the equations of order two are dependent for every form in $n > 3$ variables, they are dependent for every form in three variables.*

We shall now show that for the form in three variables the equations of order two are independent.

There are, in this case, six independent four-index symbols:

$$\begin{aligned} B_{11} = (23, 23), \quad B_{12} = (23, 31), \quad B_{13} = (23, 12), \\ B_{22} = (31, 31), \quad B_{23} = (31, 12), \\ B_{33} = (12, 12). \end{aligned} \tag{20}$$

To make the work more symmetrical we introduce, in place of the coefficients a_{ik}, the quantities, A_{ik}, obtained by dividing by a the co-factors of the elements a_{ik} in a; and, in place of the four-index symbols B_{ik}, the quantities $C_{ik} = B_{ik}/a$.

If we put

$$P_{ik} = \frac{\partial f}{\partial A_{ik}}, \quad O_{ik} = \frac{\partial f}{\partial C_{ik}}, \tag{21}$$

the equations of order two become:

$$\begin{aligned}
&2A_{11}P_{11} + A_{12}P_{12} + A_{13}P_{13} + 2C_{11}O_{11} + C_{12}O_{12} + C_{13}O_{13} = 0,\\
&2A_{21}P_{11} + A_{22}P_{12} + A_{23}P_{13} + 2C_{21}O_{11} + C_{22}O_{12} + C_{23}O_{13} = 0,\\
&2A_{31}P_{11} + A_{32}P_{12} + A_{33}P_{13} + 2C_{31}O_{11} + C_{32}O_{12} + C_{33}O_{13} = 0,\\
&A_{11}P_{12} + 2A_{12}P_{22} + A_{13}P_{23} + C_{11}O_{12} + 2C_{12}O_{22} + C_{13}O_{23} = 0,\\
&A_{21}P_{12} + 2A_{22}P_{22} + A_{23}P_{23} + C_{21}O_{12} + 2C_{22}O_{22} + C_{23}O_{23} = 0,\\
&A_{31}P_{12} + 2A_{32}P_{22} + A_{33}P_{23} + C_{31}O_{12} + 2C_{32}O_{22} + C_{33}O_{23} = 0,\\
&A_{11}P_{13} + A_{12}P_{23} + 2A_{13}P_{33} + C_{11}O_{13} + C_{12}O_{23} + 2C_{13}O_{33} = 0,\\
&A_{21}P_{13} + A_{22}P_{13} + 2A_{23}P_{33} + C_{21}O_{13} + C_{22}O_{23} + 2C_{23}O_{33} = 0,\\
&A_{31}P_{13} + A_{32}P_{23} + 2A_{33}P_{33} + C_{31}O_{13} + C_{32}O_{23} + 2C_{33}O_{33} = 0.
\end{aligned} \tag{22}$$

The following determinant, formed from the first six and last three columns of the matrix of the system, does not vanish identically:

$$D \equiv \begin{vmatrix}
2A_{11} & A_{12} & A_{13} & 0 & 0 & 0 & 0 & 0 & 0\\
2A_{21} & A_{22} & A_{23} & 0 & 0 & 0 & 0 & 0 & 0\\
2A_{31} & A_{32} & A_{33} & 0 & 0 & 0 & 0 & 0 & 0\\
0 & A_{11} & 0 & 2A_{12} & A_{13} & 0 & 2C_{12} & C_{13} & 0\\
0 & A_{21} & 0 & 2A_{22} & A_{23} & 0 & 2C_{22} & C_{23} & 0\\
0 & A_{31} & 0 & 2A_{32} & A_{33} & 0 & 2C_{32} & C_{33} & 0\\
0 & 0 & A_{11} & 0 & A_{12} & 2A_{13} & 0 & C_{12} & 2C_{13}\\
0 & 0 & A_{21} & 0 & A_{22} & 2A_{23} & 0 & C_{22} & 2C_{23}\\
0 & 0 & A_{31} & 0 & A_{32} & 2A_{33} & 0 & C_{32} & 2C_{33}
\end{vmatrix} \tag{23}$$

$$\begin{aligned}
= 32 \begin{vmatrix} A_{11} & A_{12} & A_{13}\\ A_{21} & A_{22} & A_{23}\\ A_{31} & A_{32} & A_{33} \end{vmatrix} \Bigg\{ &\begin{vmatrix} A_{21} & A_{22} & A_{23}\\ C_{21} & C_{22} & C_{23}\\ C_{31} & C_{32} & C_{33} \end{vmatrix} \begin{vmatrix} C_{31} & C_{32} & C_{33}\\ A_{21} & A_{22} & A_{23}\\ A_{31} & A_{32} & A_{33} \end{vmatrix}\\
&- \begin{vmatrix} A_{31} & A_{32} & A_{33}\\ C_{21} & C_{22} & C_{23}\\ C_{31} & C_{32} & C_{33} \end{vmatrix} \begin{vmatrix} C_{21} & C_{22} & C_{23}\\ A_{21} & A_{22} & A_{23}\\ A_{31} & A_{32} & A_{33} \end{vmatrix} .
\end{aligned} \tag{24}$$

*The equations of order two are, therefore, for the form in three variables, independent.**

From this we have, as a consequence, the

THEOREM.—*The equations of order two are all independent for the general form in $n \geqq 3$ variables.*

Consequently, from this and the lemma of §3 we have the

THEOREM.—*For the general form in $n \geqq 3$ variables all the equations of order $\mu \geqq 2$ are independent.*

Consider now the binary forms. Put

$$Z \equiv (12, 12),$$

$$(25)\quad Z_1 \equiv (12, 121) \equiv \frac{\partial (12, 12)}{\partial x_1} - (12, 12) \left\{ a_{22} a_{\underset{1}{11}} - 2 a_{12} a_{\underset{1}{12}} + a_{11} a_{\underset{1}{22}} \right\},$$

$$Z_2 \equiv (12, 122) \equiv \frac{\partial (12, 12)}{\partial x_2} - (12, 12) \left\{ a_{22} a_{\underset{2}{11}} - 2 a_{12} a_{\underset{2}{12}} + a_{11} a_{\underset{2}{22}} \right.$$

$$q \equiv \frac{\partial f}{\partial Z}, \quad q_1 \equiv \frac{\partial f}{\partial Z_1}, \quad q_2 = \frac{\partial f}{\partial Z_2}.$$

Then the equations of order two become

$$(26)\quad \begin{aligned} 2a_{11} p_{11} + a_{12} p_{12} \qquad\qquad + 2Zq &= 0, \\ 2a_{21} p_{11} + a_{22} p_{12} \qquad\qquad &= 0, \\ a_{11} p_{12} + 2a_{12} p_{22} \qquad\qquad &= 0, \\ a_{21} p_{12} + 2a_{22} p_{22} + 2Zq &= 0. \end{aligned}$$

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*Since the equations in twelve variables, which give the invariants of order two for the ternary form, are all independent, they have three common solutions, the three invariants of order two of the ternary form. CHRISTOFFEL pointed out that these may be obtained as algebraic simultaneous invariants of two quadratic forms (CHRISTOFFEL, Crelle's Journal, vol. 70 (1869), p. 65), and SOUVAROFF (Bulletin des Sciences Mathématiques et Astronomiques, vol. 4 (1873), p. 180) gave them in notation differing but slightly from that here used. The invariants are

$$H_1 = \frac{\{|C_{11} A_{22} A_{33}| + |A_{11} C_{22} A_{33}| + |A_{11} A_{22} C_{33}|\}}{|A_{11} A_{23} A_{33}|},$$

$$H_2 = \frac{\{|A_{11} C_{22} C_{33}| + |C_{11} A_{22} C_{33}| + |C_{11} C_{22} A_{33}|\}}{|A_{11} A_{22} A_{23}|},$$

$$H_3 = \frac{|C_{11} C_{22} C_{23}|}{|A_{11} A_{22} A_{33}|}.$$

The determinant of these equations is

$$(27)\qquad \begin{vmatrix} 2a_{11} & a_{12} & 0 & 2Z \\ 2a_{21} & a_{22} & 0 & 0 \\ 0 & a_{11} & 2a_{12} & 0 \\ 0 & a_{21} & 2a_{22} & 2Z \end{vmatrix} \equiv 0.$$

Hence the equations are dependent. But as the discriminant does not vanish, not all third order minors of this determinant can vanish, and hence the equations can satisfy but one linear relation.

The equations of order three are

$$(28)\qquad \begin{aligned} 2a_{11}p_{11} + a_{12}p_{12} \qquad\qquad &+ 2Zq + 3Z_1q_1 + 2Z_2q_2 = 0, \\ 2a_{21}p_{11} + a_{22}p_{12} \qquad\qquad &+ Z_2q_1 \qquad\quad = 0, \\ a_{11}p_{12} + 2a_{12}p_{22} \qquad\qquad &+ Z_1q_2 = 0, \\ a_{21}p_{12} + 2a_{22}p_{22} + 2Zq + 2Z_1q_1 &+ 3Z_2q_2 = 0. \end{aligned}$$

The following determinant of the matrix does not identically vanish:

$$\begin{vmatrix} a_{12} & 2Z & 3Z_1 & 2Z_2 \\ a_{22} & 0 & z_2 & 0 \\ a_{11} & 0 & 0 & z_1 \\ a_{21} & 2Z & 2Z_1 & 3Z_2 \end{vmatrix} = 2Z(a_{11}Z_2^2 - a_{22}Z_1^2).$$

Hence the equations are all independent.

We have therefore the following

THEOREM.—*If* $n \geqq 3$, $\mu \geqq 2$, *or* $n = 2$, $\mu \geqq 3$, *the equations determining the invariants of order* μ *are, for the general quadratic differential form in* n *variables, independent, and hence the number of such invariants is given by the excess of the number of the variables* p *in these equations, over the number of the equations themselves.*

§ 6. *Equations of orders zero and unity.*

The equations of order zero may be written

$$(1)\qquad \sum_{\lambda=1}^{n}(1 + \epsilon_{i\lambda})a_{r\lambda}p_{i\lambda} = 0 \qquad (r, i = 1, 2, \cdots, n),$$

where

$$\epsilon_{i\lambda} = 0, \quad \text{if} \quad i \neq \lambda; \quad \epsilon_{ii} = 1.$$

There are n^2 of these equations. They contain $n(n+1)/2$ variables p_{ik}. It can readily be shown, by taking the n equations characterized by the index $i=1$, $n-1$ equations characterized by $i=2$, and so on, and finally, by taking one equation characterized by $i=n$, that we can form a set of $n(n+1)/2$ equations whose determinant does not vanish. This selection is made possible by the fact that since $a \not\equiv 0$ not all the minors formed from any k columns can vanish identically. Hence we have the

THEOREM.—*The equations of order zero include as many independent equations as they have variables, and consequently have no common solutions.*

In passing to the equations of order unity, $n^2(n+1)/2$ new variables and the same number of equations, are added. The equations added are the final equations of order unity, and are, as we have seen, all independent. Hence, by the same reasoning as that of § 3, the system of $n^2 + n^2(n+1)/2$ equations in

$$\frac{n(n+1)}{2} + \frac{n^2(n+1)}{2} = N_{n1}$$

variables contains N_{n1} independent equations. We have, then, the

THEOREM: *The equations of order unity include as many independent equations as they have variables and hence have no common solutions.*

This is, of course, RICCI'S result, but it is obtained in a wholly different way.

§ 7. *Number of invariants.*

The number of variables, $N_{n\mu}$, in the equations of order μ is the number of the quantities a_{ik} and of their derivatives of all orders not exceeding μ. The number of equations, $M_{n\mu}$, is the number of derivatives of the quantities ξ_r of all orders not exceeding $\mu+1$.

We have *

$$N_{n\mu} = \frac{n(n+1)}{2}\frac{(n+\mu)!}{\mu!\,n!}, \tag{1}$$

$$M_{n\mu} = n\left\{\frac{(n+\mu+1)!}{n!\,(\mu+1)!} - 1\right\}. \tag{2}$$

The difference,

$$J_{n\mu} \equiv N_{n\mu} - M_{n\mu} = n + \left\{\frac{(n-1)(\mu-1)-2}{2}\right\}\frac{(n+\mu)!}{(n-1)!\,(\mu+1)!}, \tag{3}$$

is, if the equations of order μ are all independent, the number of invariants of order not greater than μ.

* Cf. LEVI-CIVITA, R. Ist. Veneto, Atti, ser. 7, vol. 5_2 (1894), p. 1466–8.

If the equations of order $\mu - 1$ are also independent there are $J_{n,\mu-1}$ invariants of order not greater than $\mu - 1$. Hence there are

$$I_{n\mu} \equiv J_{n\mu} - J_{n,\mu-1} = n\frac{\mu-1}{2}\frac{(n+\mu-1)!}{(n-2)!\,(\mu+1)!} \tag{4}$$

invariants of order μ.

Since neither the equations of order zero nor those of order unity have any common solutions, and since the equations of order two are, if $n > 2$, all independent, there are

$$J_{n2} = \frac{(n-2)(n-1)\,n\,(n+3)}{12} \tag{5}$$

invariants of order two.

Since for the binary form there are

$$J_{23} = 2 \tag{6}$$

invariants of order not higher than three and since the four equations of order two contain four variables and include but three independent equations, one of these invariants is of order two and one of order three. For all other values of n and μ the equations of order $\mu - 1$ as well as those of order μ are all independent and hence there are $I_{n\mu}$ invariants.

We have, then, the following

THEOREM: *The number of invariants of order μ for the general quadratic form in n variables is*

$$I_{n\mu} = n\frac{\mu-1}{2}\frac{(n+\mu-1)!}{(n-2)!\,(\mu+1)!}$$

provided $n > 2$, $\mu > 2$, or $n = 2$, $\mu > 3$.

If $n = 2$ and $\mu = 2$ or $\mu = 3$, there is one invariant.

If $n > 2$ and $\mu = 2$, there are

$$J_{n2} = \frac{(n-2)(n-1)\,n\,(n+3)}{12}$$

invariants.

If $\mu = 0$ or $\mu = 1$, there are no invariants.

§ 8. *Simultaneous invariants.*

In studying those quadratic differential forms ϕ, in n variables, which can be expressed as the sum of $n + 1$ differential squares, it is found that to the form ϕ a second form ψ, also in n variables, can be adjoined, and the theory of the simultaneous invariants of these two forms is important in the study of the geo-

metric interpretation of ϕ as the squared linear element of an n-dimensional surface in a euclidean space of $n+1$ dimensions. For $n=2$, ϕ and ψ are the first and second fundamental forms of a surface, and the simultaneous invariant H, of order zero, is the mean curvature.

The invariants of order two of a ternary quadratic differential form can be expressed as the simultaneous invariants of order zero of two such forms.

The simultaneous invariants of two or more quadratic differential forms may be found by extending the group to the coefficients of those forms and to their derivatives. The equations determining the invariants may be obtained from those for a single form by annexing to them a set of terms constructed for each additional form in the same way as the terms already found for a single form. It is evident that if the equations are, for a single form, independent, they will be so for several forms.

From m forms we obtain then $m N_{n\mu}$ variables and $M_{n\mu}$ equations. Hence if the equations are, for a single form, all independent there will be

$$m N_{n\mu} - M_{n\mu} \tag{1}$$

invariants of order μ or lower. But of these

$$m(N_{n\mu} - M_{n\mu}) \tag{2}$$

are invariants of the m separate forms. There remain

$$S_{m\,n\,\mu} = (m-1) M_{n\mu} \tag{3}$$

true simultaneous invariants of the m forms. If the same conditions hold for the order $\mu - 1$ there will be

$$\begin{aligned} I_{m\,n\,\mu} = S_{m\,n\,\mu} - S_{m\,n\,\mu-1} &= (m-1)(M_{n\mu} - M_{n\,\mu-1}) \\ &= (m-1)\frac{(n+\mu)!}{(n-1)!\,(\mu+1)!} \end{aligned} \tag{4}$$

true simultaneous invariants of order μ for the m forms. If $m=2$,

$$I_{2n\mu} = \frac{(n+\mu)!}{(n-1)!\,(\mu+1)!}. \tag{5}$$

Hence the $(m-1) I_{2n\mu} = I_{2n\mu}$ true simultaneous invariants of m points may be taken as invariants of the $m-1$ pairs of forms obtained by associating every given form with the remaining $m-1$ in succession.

When the number of forms is greater than unity, invariants of order zero and unity also exist.

A detailed consideration of the special cases leads us to the following

THEOREM: *The number of true simultaneous invariants of order zero is, for m forms,*

$$m\frac{n(n+1)}{2}-n^2.$$

If $n=2$ and $\mu=3$, or $\mu=2$ and $n>2$, the number of true simultaneous invariants of order μ is, for m forms,

$$(m-1)\frac{n(n+\mu)!}{(\mu+1)!\,(n-1)!}+\frac{mn(n-1)}{2}.$$

For all other values of n and μ the number of true simultaneous invariants of order μ is

$$I_{mn\mu}=(m-1)\frac{n(n+\mu)!}{(\mu+1)!\,(n-1)!},$$

and the invariants can be taken as invariants of pairs of forms.

HARVARD UNIVERSITY,
June 1, 1901.

ON THE NATURE AND USE OF THE FUNCTIONS EMPLOYED IN THE RECOGNITION OF QUADRATIC RESIDUES*

BY

EMORY McCLINTOCK

INTRODUCTION.

The congruence $n \equiv x^2 (\text{mod } k)$ is possible, and n is therefore a quadratic residue of k, when n is a quadratic residue of each prime factor of k, so that in order to determine the possibility of the congruence in all cases we must be able to determine its possibility when k is any prime number. The case $k = 2$ is simple, but when k is an odd prime the problem presents some difficulties, and it has perhaps received more attention than any other in the theory of numbers. LEGENDRE introduced the symbol $(n/k) = \pm 1 \equiv n^{\frac{1}{2}(k-1)} (\text{mod } k)$, the sign being $+$ or $-$ as n is or is not a quadratic residue of the prime number k, and since his time the problem has consisted in determining the sign of (n/k) for any given values of n and k, n being prime to the odd prime k. The method of evaluation, or algorithm, of LEGENDRE, improved by JACOBI, is still the standard solution. It requires the use of the *law of quadratic reciprocity* formulated by LEGENDRE, though perceived earlier by EULER: *theorema fundamentale*, as it was called by GAUSS, who first supplied for it a satisfactory demonstration. The derivation of this law has attracted unusual attention from many mathematicians, eight demonstrations having been produced by GAUSS alone. The chief improvement since the time of JACOBI consists in an observation made independently by SCHERING and KRONECKER,† namely, that "GAUSS's characteristic," μ, is available for the proof of the law of reciprocity when k is not prime. The definition $(n/k) = (-1)^{\mu}$, employed by TANNERY in his proof of the usual algorithm, is one of two employed in the present paper, and is herein extended and applied to wider purposes, with only the slightest reference to the law of reciprocity. I find great advantage in substituting for the symbol μ the broader symbol $\mu(n, k)$, so as to be able to discuss the function μ for different values of n and k, and thereby to develop relations of the functions $\mu(n, k)$ im-

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† See TANNERY, *Leçons d'Arithmétique* (Paris, 1894); BACHMANN, *Elemente der Zahlentheorie* (Leipzig, 1892); BAUMGART, *Ueber das quadratische Reciprocitätsgesetz* (Göttingen, 1885). These works will be referred to hereafter by the names of the authors.

mediately corresponding to certain elementary relations of the functions (n/k) which are previously set forth from another point of view. The elementary relations supply a simple algorithm for the evaluation of (n/k) without requiring the use of the law of reciprocity. Systems have been proposed in the past for the purpose of avoiding the use of that law, EISENSTEIN's being the best; and a comparison of the new algorithm is therefore desirable, both with that heretofore customary and with that of EISENSTEIN. This paper therefore comprises the following subjects:

I. The function (n/k) and its elementary relations;
II. Theorems, underlying the elementary relations, concerning $\mu(n, k)$;
III. The new algorithm derived from the elementary relations, and
IV. Comparison of the new algorithm with previous methods.

I. THE FUNCTION (n/k) AND ITS ELEMENTARY RELATIONS.

1. Consider the following table of two arguments, n and k, the latter odd:

	$n=1$	2	3	4	5	6	7	8	9
$k=1$	+	+	+	+	+	+	+	+	+
3	+	−		+	−		+	−	
5	+	−	−	+		+	−	−	+
7	+	+	−	+	−	−		+	+
9	+	+		+	+		+	+	
11	+	−	+	+	+	−	−	−	+
13	+	−	+	+	−	−	−	−	+
15	+	+		+			−	+	
17	+	+	−	+	−	−	−	+	+

It is constructed by writing a row and a column alternately in accordance with certain rules, and is assumed to be continued indefinitely. Beginning with + for $n=k=1$, the row $k=1$ is written by repeating + throughout, and the column $n=1$ in the same way. The row $k=3$ has one place thus filled. We fill one place adjacent by changing the sign, and we leave the third (kth) place vacant. The row is then written off in periods of three places by repetition. The column $n=2$ has now two places filled, + −. We fill two more places by repetition with change of sign, and then write off the column by copying periods of four places. The row $k=5$ has now two places filled, + −. We repeat these in inverse order, − +, without change of sign, thus completing four places, leaving the fifth (kth) place vacant, and go on to complete the row in periods of five places by repetition. The column $n=3$ has now three places supplied, say $+\ v\ -$, where v means vacancy. Three more places are filled by copying in direct order with change of sign, $-\ v\ +$, and the column is written off in periods of six places. The row $k=7$ has now three places filled, + + −,

and three more are obtained in inverse order with change of sign, $+\ -\ -$, the rest following in periods of seven, the seventh (kth) place being left vacant. It is now possible to perceive the rules by which the table may be extended without limit, and by which every place shall be occupied by $+$, $-$, or a regular vacancy. For beginning each row, we find $\frac{1}{2}(k-1)$ places already written, belonging to columns previously constructed, and we obtain as many more by copying in inverse order, with change of sign if $\frac{1}{2}(k-1)$ is odd, and the kth place is left vacant; and the rest of the row is written off by repetition, in periods of k places each. For beginning each column, we find n places already written, belonging to rows previously constructed, and we obtain as many more by copying in direct order, with change of sign if n or $n-1$ is of the form $4r+2$; and the rest of the column is written off by repetition, in periods of $2n$ places each. The vacancies occur necessarily wherever n and k contain a common factor, and nowhere else: explicitly if n is a multiple of k, and as a consequence of the mode of writing the rows and columns if both n and k are multiples of some odd prime, say p; because in row p, column p, there is a primary vacancy, which by the rules of formation is repeated for every period of p places throughout the rows and columns, including the row $xp = k$ and the column $yp = n$.

2. The columns may be extended upwards by copying in periods of $2n$ places each, so that k may be negative, and the rows may be extended to the left by copying in periods of k places each, so that n may be negative, but the process cannot be extended so as to make both negative, for the rules of formation are not symmetrical. For example, the place where $k=-5$ and $n=1$ is a place where $+$ must be written, by repetition from $k=1$, $n=1$; and from this, by row-repetition, we might write $+$ in the place where $k=-5$, $n=-4$. But since we have $-$ for $k=11$, $n=7$, we must repeat $-$ for $k=11$, $n=-4$, and from this, by column-repetition, we might write $-$ in the place where $k=-5$, $n=-4$, contrary to the former result.

3. The signs entered in the table, $+$ and $-$, are abbreviations of $+1$ and -1, and these quantities are functions of the integers n and k, the latter being always odd and prime to n. Let the function in question be denoted by (n/k). Then $(1/1)=1$. From the rules of formation we have, when $n<k$, both being positive,

$$(k-n/k) = (-1)^{\frac{1}{2}(k-1)}(n/k); \tag{1}$$

again, for all values of n, with k positive and x integral,

$$(n/k) = (n-xk/k); \tag{2}$$

and finally, for all values of k, with n positive,

$$(n/k) = (-1)^{\frac{1}{2}xn(n-1)}(n/k-2xn). \tag{3}$$

These three equations, with $(1/1) = 1$, supply the complete definition of the function (n/k), as has been shown in paragraph 1. Three other important elementary relations exist, namely,

$$(4) \qquad (2n/k) = (-1)^{\frac{1}{2}n(k-1)}(2n/k - 2n),$$

$$(5) \qquad (k'/k) = (-1)^{\frac{1}{4}(k-1)(k'-1)}(k/k'),$$

$$(6) \qquad (m/k)(n/k) = (mn/k).$$

4. For the proof of (4), these lemmas are useful:

$$(A) \qquad (2/k) = (-1)^{\frac{1}{2}c(c+1)} = (-1)^{\frac{1}{8}(k^2-1)}, \qquad \text{where} \qquad k = 2c + 1,$$

$$(B) \qquad (n/2n - k) = (-1)^{\frac{1}{2}n(n-1)}(n/k),$$

$$(C) \qquad (2/k)(n/k) = (2n/k).$$

Pursuing the course of paragraph 1 regarding the column $n = 2$, we obtain (A) by observing that $(2/1) = 1$ follows by (2) from $(1/1) = 1$, and that $(2/3) = -1$ follows by (1) from $(1/3) = 1$, which again comes by (3) from $(1/1) = 1$; after which, by (3), $(2/k) = (-1)^x(2/k - 4x)$, so that if $k = 4x + 1 = 2c + 1$, c is even and $(2/k) = (-1)^{\frac{1}{2}c}$, while if $k = 4x + 3 = 2c + 1$, c is odd and $(2/k) = (-1)^{\frac{1}{2}(c-1)}(2/3) = (-1)^{\frac{1}{2}(c+1)}$, both results combined forming (A). To obtain (B) we have

$$(k + m/k) = (m/k) = (-1)^{\frac{1}{2}m(m-1)}(m/k + 2m)$$
$$= (-1)^{\frac{1}{2}m(m-1)}(-1)^{\frac{1}{2}(k+2m-1)}(k + 2m - m/k + 2m).$$

Writing $n - k$ for m and remarking, concerning the exponents of -1, that $k^2 - 1$ and $2k - 2$ are divisible by 4, we have (B). To prove (C), we remark that it is true when $k = 1$, $k = 3$, etc., and we need to show that it is true for any given value of k provided it is true for all lower values. First, if $k > 2n$, and if k' is the positive value of $\pm(k - 4n)$, we have $k' < k$, and, by (3) and (B), $(n/k) = (n/k')$; also, $(2n/k) = (-1)^n(2n/k')$. By hypothesis, $(2n/k') = (2/k')(n/k')$, and, by (A), $(2/k') = (-1)^{\frac{1}{8}(k^2-1-8n)}$, since $(-1)^n = (-1)^{nk}$, so that $(2n/k) = (-1)^{\frac{1}{8}(k^2-1)}(n/k) = (2/k)(n/k)$. Secondly, if $k < 2n$, let

$$n = mk \pm n',$$

where m is a positive integer and $n' < \frac{1}{2}k$. If the sign is $+$, we have, by (2), $(n'/k) = (n/k)$ and $(2n'/k) = (2n/k)$, while if the sign is $-$, we have, by (1), $(n'/k) = (-1)^{\frac{1}{2}(k-1)}(n/k)$ and $(2n'/k) = (-1)^{\frac{1}{2}(k-1)}(2n/k)$; and we are thus enabled to deal with n' as in the first case we dealt with n.

5. We may now prove (4) by observing that, by (3),

$$(n/k - 2n) = (-1)^{-\frac{1}{2}n(n-1)}(n/k),$$

and, by (A), $(2/k-2n)=(-1)^{\frac{1}{2}n(n-k)}(2/k)$; which being combined by (C) we have $(2/k-2n)(n/k-2n)=(2n/k-2n)=(-1)^{\frac{1}{2}n(1-k)}(2n/k)$, or

$$(4) \qquad (2n/k)=(-1)^{\frac{1}{4}n(k-1)}(2n/k-2n);$$

from which, again, the corollary (5) immediately follows. For

$$(2n/k)=(-1)^{\frac{1}{4}(k-1)}(k-2n/k)$$

and $(2n/k-2n)=(k/k-2n)$, whence $(k-2n/k)=(-1)^{\frac{1}{4}(k-1)(n-1)}(k/k-2n)$. If in this we write k' for $k-2n$, multiplying the second member by $(-1)^{\frac{1}{4}(k^2-1)}=1$ and by $(-1)^{-n(k-1)}=1$, we have

$$(5) \qquad (k'/k)=(-1)^{\frac{1}{4}(k-1)(k'-1)}(k/k').$$

This is in fact the well known law of reciprocity, though it is not yet proved that (n/k), as here already defined, is the (n/k) to which the law of reciprocity has heretofore applied. What is needed for this purpose is proof that $(n/k)=1$ when n is a quadratic residue of k, or, more briefly, that

$$(D) \qquad (n^2/k)=1.$$

See paragraph 21.

6. That (6) becomes (D) when $m=n$ is obvious, since $(n/k)^2=(\pm 1)^2=1$. In view of (2), it is sufficient to prove (6) when both m and n are less than k. We observe that $(m/k)(n/k)=(mn/k)$ is true for the smallest values of k, and we need to show that it is true for any given value of k if it is true for all smaller values. Let us consider that mn is even, namely, $m=2^pa$ and $n=2^qb$, where p or q may be 0, and $p+q=r$, say $mn=2^rab$, where $r>0$ and ab is odd; for if not, m may be replaced by $m'=k-m$, even, because $(m/k)=(-1)^r(m'/k)$ and $(mn/k)=(-1)^r(m'n/k)$, by (1). Since, by (C),

$$(2^pa/k)=(2^p/k)(a/k),\ (2^qb/k)=(2^q/k)(b/k),\ (2^p/k)(2^q/k)=(2^r/k),$$

and

$$(2^rab/k)=(2^r/k)(ab/k),$$

it is sufficient to prove that $(ab/k)=(a/k)(b/k)$. Let $2^rab\pm d=ek$, where d and e are odd and less than k; so that $2^rab\equiv\mp d \pmod{k}$, $ek\equiv\pm d \pmod{a}$, etc. Let us assume for the present that d is prime to e, and therefore that both are prime to a and b. Let d have first the negative sign, so that

$$2^rab-d=ek;$$

then

$$(2^rab/k)=(d/k),\ (2^rab/e)=(d/e),\ (2^rab/d)$$
$$=(ek/d),\ (ek/a)=(-d/a),\ (ek/b)=(-d/b),$$

whence—the new letters indicating powers of -1 in accordance with (1), (5). (A), and (C)—

$$(ab/k) = (2^r/k)(d/k) = \text{A}(d/k), \quad (d/k) = \text{B}(k/d) = \text{BC}(e/d)(d/a)(d/b),$$
$$(e/d) = \text{D}(e/a)(e/b), \quad (a/k) = \text{E}(k/a) = \text{EF}(e/a)(d/a),$$
$$(b/k) = \text{G}(k/b) = \text{GH}(e/b)(d/b),$$

and

$$(ab/k) = \text{ABCD}(d/a)(d/b)(e/a)(e/b) = \text{EFGH}(d/a)(d/b)(e/a)(e/b) = (a/k)(b/k),$$

as required, provided ABCDEFGH $= 1$. Let $a = 2\alpha + 1$, $b = 2\beta + 1$, $d = 2\delta + 1$, $e = 2\epsilon + 1$, besides $k = 2c + 1$, as before; then the exponent of -1 in A is $\frac{1}{2}rc(c+1)$, in B is $c\delta$, in C is $\frac{1}{2}r\delta(\delta+1) + \delta(\alpha+\beta)$, in D is $\frac{1}{2}r\epsilon(\epsilon+1) + \epsilon(\delta+\alpha+\beta)$, in E is $c\alpha$, in F is α, in G is $c\beta$, and in H is β, and what is to be proved is the congruence

$$(\alpha+\beta)(\delta+1+c+\epsilon) + \delta(c+\epsilon) + \tfrac{1}{2}r(c^2+c+\delta^2+\delta+\epsilon^2+\epsilon) \equiv 0 \pmod 2.$$

Towards proving this we have $2^r ab - d - ek = 2^r(4\alpha\beta + 2\alpha + 2\beta + 1) - 2\delta - 2 - 4c\epsilon - 2c - 2\epsilon = 0$. From this, if $r = 1$, we have $\delta + c + \epsilon \equiv 0 \pmod 2$, and $\alpha + \beta + c\epsilon + \frac{1}{2}(\delta + c + \epsilon) \equiv 0$; and by combining these with the congruence in question, the latter is reduced to an identity, $\frac{1}{2}(\delta + c + \epsilon)^2 \equiv 0 \pmod 2$. If r is even, $\delta + 1 + c + \epsilon \equiv 0 \pmod 2$, which again reduces the congruence to an identity, $\delta(c+\epsilon) \equiv \delta(\delta+1) \equiv 0 \pmod 2$. Finally, if r is greater than 2, and odd, $\delta + 1 + c + \epsilon \equiv 0 \pmod 2$, and $c\epsilon + \frac{1}{2}(\delta + 1 + c + \epsilon) \equiv 0$, by which the congruence is reduced to $\frac{1}{2}(\delta + 1 + c + \epsilon)^2 \equiv 0 \pmod 2$. We have hitherto taken d, in $2^r ab \pm d = ek$, with the negative sign: if it be positive, changes must be made to correspond, so that F and H disappear, while A and D are increased respectively by c and ϵ, the required congruence thus becoming

$$(\alpha+\beta)(\delta+c+\epsilon) + (\delta+1)(c+\epsilon) + \tfrac{1}{2}r(c^2+c+\delta^2+\delta+\epsilon^2+\epsilon) \equiv 0 \pmod 2.$$

The given equation is now $2^r(4\alpha\beta + 2\alpha + 2\beta + 1) + 2\delta - 2c - 2\epsilon - 4c\epsilon = 0$. If $r = 1$, $\delta - c - \epsilon + 1 \equiv 0 \pmod 2$, and $\alpha + \beta + c\epsilon + \frac{1}{2}(\delta - c - \epsilon + 1) \equiv 0$, producing the identity $\frac{1}{2}(\delta - c - \epsilon + 1)(\delta - c - \epsilon - 1) \equiv 0$; if r is even, $\delta + c + \epsilon \equiv 0 \pmod 2$, producing $(\delta+1)(c+\epsilon) \equiv -\delta(\delta+1) \equiv 0$; and if r is odd and greater than 2, $\delta + c + \epsilon \equiv 0 \pmod 2$, and $c\epsilon + \frac{1}{2}(c + \epsilon - \delta) \equiv 0$, producing $\frac{1}{2}(\delta + c + \epsilon)^2 \equiv 0$. We have thus far assumed that d, in $mn \pm d = ek$, has no common factor with c, which if mn is even is necessarily true only when $mn < 2k$; and have proved, on that assumption, that $(m/k)(n/k) = (mn/k)$. Let $mt_1 = m'$, $nt_2 = n'$, and $t_1 t_2 = t$; t_1 and t_2 being odd and prime to k, and such that m', n', $dt = d'$, and $et = e'$, are all respectively less than k. In the more general expression $m'n' \pm d' = e'k$, d' is not prime to e', but it is now easy to see that

$$(m'/k)(n'/k) = (mt_1/k)(nt_2/k) = (m/k)(n/k)(t_1/k)(t_2/k) = (mn/k)(t_1 t_2/k)$$
$$= (\mp d/k)(t/k) = (\mp dt/k) = (\mp d'/k) = (m'n'/k);$$

because $2mt_1$, $2nt_2$, $2t_1t_2$, and $2dt$ are all respectively less than $2k$, and each of them may be represented by $2ab$ in $2ab \pm f = k$, the terms of which are prime to each other.

II. Theorems, underlying the elementary relations, concerning $\mu(n, k)$.

7. Let n and c be positive integers, and let $k = 2c + 1$ be prime to n. Let the number of negative absolutely smallest residues in the series n, $2n$, $\cdots$, $cn \pmod k$ be denoted by $\mu(n, k)$, and let $\mu(n, 1) = 0$. The complementary series $(c+1)n$, $(c+2)n$, $\cdots$, $2cn \pmod k$, or $(k-c)n$, $(k-c+1)n$, $\cdots$, $(k-1)n \pmod k$, comprises the same residues in reverse order, with change of sign. The terms n, $2n$, $\cdots$, $2cn$ being incongruent, modulo k, all the residues of both series, $2c$ in number, are different, comprising the numbers, in some order, from $-c$ to $+c$ inclusive; those of the first series comprising, in some order, every number from 1 to c, each affected by either the positive or the negative sign, and those of the second series comprising the same numbers in reverse order, each with a change of sign. The number of negative residues in the first series being $\mu(n, k)$, that in the complementary series is, therefore, $c - \mu(n, k)$; or, if $n < k$, since $(k-r)n \equiv r(k-n) \pmod k$,

$$\mu(k-n, k) = c - \mu(n, k) \equiv \tfrac{1}{2}(k-1) + \mu(n, k) \pmod 2. \tag{7}$$

Had $n \pm xk$, positive, with x integral, been taken instead of n in the first series, the residues would have been the same, so that

$$\mu(n, k) = \mu(n \pm xk, k). \tag{8}$$

Let β, γ, δ be positive integers not greater than c, and let the residue $\beta m \pmod k$ be γ or $-\gamma$, and let $\gamma n \pmod k$ be δ or $-\delta$. Let the number of values of β for which $\beta m \equiv \gamma$, where γ is such that $\gamma n \equiv -\delta$, be denoted by r; for which $\beta m \equiv -\gamma$ when $\gamma n \equiv \delta$, by s; and for which $\beta m \equiv -\gamma$ when $\gamma n \equiv -\delta$, by t. Then in $r + s$ cases we shall have $\beta mn \equiv -\delta$, negative, so that $r + s = \mu(mn, k)$; also, $s + t = \mu(m, k)$, and $r + t = \mu(n, k)$; whence

$$\mu(mn, k) = \mu(m, k) + \mu(n, k) - 2t \equiv \mu(m, k) + \mu(n, k) \pmod 2. \tag{9}$$

8. A second modulus, $k + 2n$, is, with k, prime to n. If n is even, say $n = 2r$, the last residue of the series n, $2n$, $\cdots$, $(c+n)n \pmod{k+2n}$ is $2r(c+2r) - r(2c+1+4r) = -r$, negative, and if n is odd, say $n = 2r+1$, the last residue is $(2r+1)(c+2r+1) - r(2c+4r+3) = c + r + 1$, a positive quantity. The form of each residue being $an - b(k+2n)$, b must have r values, from 1 to r, for each of which there is one negative residue whose value is from $-c-1$ to $-c-n$ inclusive; because, for any given value of b, say $b = g$, there cannot be more than one such residue, since however small al-

gebraically the first may be, say $-c-n$, the next must be $-c$, outside of the limit; and there must be one, for if not, we shall have two adjacent residues, say $an-(g-1)(k+2n)$ and $(a+1)n-(g+1)(k+2n)$, differing in value by as much as $2k+3n$, which is absurd. In addition to the r negative values smaller algebraically than $-c$, there are in the series before us $\mu(n, k)$ negative residues, with values from -1 to $-c$, exactly the same as in the original series $n, 2n, \cdots, cn \pmod k$; for each residue in the original series, say $a=an-bk$, where $b<\frac{1}{2}(n+1)$, since $a<\frac{1}{2}k$, is represented in the larger series by $a=(a+2b)n-b(k+2n)$. Hence, whether $n=2r$ or $n=2r+1$,

$$\mu(n, k+2n)=\mu(n, k)+r. \tag{10}$$

9. All the numbers from 1 to c, each either positive or negative, are residues of the series $2n, 2\cdot 2n, \cdots, c\cdot 2n \pmod k$, where $k=2c+1$, being of the form $a\cdot 2n-bk$, where $b<n+1$ because $a<\frac{1}{2}k$, and are repeated identically as residues of the form $(a+b)2n-b(k+2n)$ in the series

$$2n, 2\cdot 2n, \cdots, (c+n)2n \pmod{k'},$$

where $k'=k+2n$. Every other negative residue of the latter series, say, $a=a\cdot 2n-bk'<-c$, where $a<c+n+1$ and $b<n+1$, is one of a pair, the other being $a'=a'\cdot 2n-b'k'$, where $a'=c+n+1-a$ and $b'=n+1-b$, the sum being $a+a'=-2c-n-1$; unless $a'=\frac{1}{2}(c+n+1)=a$ and $b'=\frac{1}{2}(n+1)=b$, which can occur only when n and $c+1$ are both odd, $c+1$ being $\frac{1}{2}(k+1)$.* Hence,

$$\mu(2n, k')\equiv\mu(2n, k)+\tfrac{1}{2}n(k+1) \pmod 2. \tag{11}$$

By (7) and (8), $\mu(2n, k')\equiv\mu(k, k')+\frac{1}{2}(k'-1) \pmod 2$, and $\mu(2n, k)=\mu(k', k)$, while $\frac{1}{2}n(k+1)-\frac{1}{2}(k'-1)=\frac{1}{2}(k-1)(n-1)\equiv\frac{1}{4}(k-1)(k'-1) \pmod 2$, since $\frac{1}{4}(k^2-1)\equiv 0$, so that

$$\mu(k, k')\equiv\mu(k', k)+\tfrac{1}{4}(k-1)(k'-1) \pmod 2.\dagger \tag{12}$$

* Similarly, every positive residue greater than c is one of a pair, say $a=a\cdot 2n-bk'$ and $a'=(c+n-a)2n-(n-1-b)k'$, the sum being $a+a'=2c+n+1$; with one exception when n and c, or $\frac{1}{2}(k-1)$, are both odd.

† Of the theorems (7) to (12), that numbered (8) is little more than the formulation of a truism; a formula equivalent to a special case of (10), wherein n is odd and n and k are both prime, appears as a step in BUSCHE'S proof of the law of reciprocity (see BAUMGART'S collection of proofs); and (12) is new only in its derivation, being well-known in certain demonstrations of that law, the quantity $\mu(k', k)$ being denoted by ν. The best demonstration is that of ZELLER, which might be incorporated in the present theory if there were need of an independent proof of (12) distinct from that of (11), of which (12) is an immediate corollary. ZELLER'S method, even as recently reproduced by BAUMGART and by BACHMANN, is confined needlessly to purely prime values of k and k'.

10. That the definition $\mu(n, 1) = 0$ is not arbitrary may be seen upon substituting 1 for k and 0 for $\mu(n, k)$ in the congruences numbered (7) to (12), causing both sides of each to vanish identically: both sides of (7), (8) and (9), obviously; of (10), because $\mu(n, 2n + 1) = r$, since the series $n, 2n, \cdots, nn \pmod{2n + 1}$ supplies negative residues only in the even terms, r in number; of (11), because $\mu(2n, 2n + 1) = n$, by (7); and of (12), because $\mu(1, k') = 0$, there being no negative residues in the series $1, 2, \cdots, \frac{1}{2}(k' - 1) \pmod{k'}$.

11. The congruences relating to μ are, as has been seen, derived readily from the definition of μ, but the properties expressed by them are not all independent. We have seen that the sixth, numbered (12), follows at once from (7), (8) and (11). It may be shown that independent proof is required for three only, provided the three are rightly chosen. One combination which may serve as a basis for the others comprises (8), (9) and (10). Assuming these to be known, we have from (9)

$$\mu(n^2, k) \equiv 0 \pmod 2. \tag{13}$$

By (10) and (8), when m is even,

$$\mu(m, k) = \mu(m, k + 2m) - \tfrac{1}{2}m = \mu(m, k + m^2) - \tfrac{1}{2}m = \mu(m + k + m^2, k + m^2) - \tfrac{1}{2}m.$$

If we put $m = k - 1$, we have $m + k + m^2 = k^2$, and, observing (13), we have

$$\mu(k - 1, k) \equiv \tfrac{1}{2}(k - 1) \pmod 2; \tag{14}$$

Since $\mu(k - n, k) = \mu(kn - n, k) \equiv \mu(k - 1, k) + \mu(n, k) \pmod 2$, by (9), we have now the proof of (7). By (10), $\mu(2, 1 + 4) = \mu(2, 1) + 1 = 1$, and in general, $\mu(2, 1 + 4m) = m$, or $\mu(2, k) = \frac{1}{2}c$ if c is even; also, by (7), $\mu(2, 3) = 1$ since $\mu(1, 3) = 0$ and $\frac{1}{2}(3 - 1) = 1$, and as before,

$$\mu(2, 3 + 4m) = 1 + m,$$

or $\mu(2, k) = \frac{1}{2}(c + 1)$ if c is odd. (Both of these results might be had directly from the series $2, 4, \cdots, 2c$, modulo k.) Hence

$$\mu(2, k) \equiv \tfrac{1}{2}c(c + 1) \equiv \tfrac{1}{8}(k^2 - 1) \pmod 2. \tag{15}$$

Therefore $\mu(2, k + 2n) \equiv \mu(2, k) + \frac{1}{2}n(n + k) \pmod 2$, and by (10) $\mu(n, k + 2n) \equiv \mu(n, k) - \frac{1}{2}n(n - 1) \pmod 2$. Combining these, by (9), we have $\mu(2, k + 2n) + \mu(n, k + 2n) \equiv \mu(2n, k + 2n) \equiv \mu(2n, k) + \frac{1}{2}n(k + 1)$, which proves (11).

12. Similarly, we may derive (7) and (10) from (8), (9) and (11), assumed to be known. From (11), by repetition, $\mu(2, k + 4) \equiv \mu(2, k) + 1 \pmod 2$;

also, $\mu(2, 3) = 1$, since $\mu(2, 1) = 0$. We thus obtain (15), and may now derive (10) from (11) by reversing the process of paragraph 11, for odd values of n; (10) being derived from (11), for even values of n, by repetition. After this, we may obtain (7) as in the preceding paragraph.

13. Let

$$(n/k) = (-1)^{\mu(n,k)}. \tag{16}$$

Under this definition the elementary relations numbered (1), (2), (6) and (5) are at once proved by the μ-theorems (7), (8), (9) and (12) respectively. Let the μ-theorem (10) be applied x times, with $k - 2n$ substituted for k; then, since $(-1)^{\frac{1}{2}n(n-1)} = (-1)^r$, whether $n = 2r$ or $n = 2r + 1$, we have the elementary relation numbered (3). If in the μ-theorem (11) we write k for k' and $k - 2n$ for k, we shall have the proof of the elementary relation numbered (4), since $\frac{1}{2}n(k - 2n + 1) \equiv \frac{1}{2}n(k - 1) \pmod 2$. Of these six elementary relations (1), (2), (6) and (5) are known, the latter being the law of reciprocity. It is to be noted that (1) and (4) may be derived from (2), (3) and (6), as has been shown of the corresponding μ-theorems in paragraph 11, or (1) and (3) from (2), (4) and (6), corresponding to paragraph 12. The proof of the law of reciprocity, (5), as a corollary of (4), if we know (1) and (2), has been given in paragraph 5, under the form

$$(k - 2n/k) = (-1)^{\frac{1}{4}(k-1)(n-1)}(k/k - 2n). \tag{17}$$

III. The new algorithm, and its derivation from the elementary relations.

14. When, in (n/k), either n or k is divisible by a square number, the quotient may be put in its place. For, since $(m/k) = \pm 1$, $(m/k)^2 = 1 = (m^2/k)$, by (6), so that

$$(m^2n/k) = (n/k). \tag{18}$$

Also, that

$$(n/h^2k) = (n/k) \tag{19}$$

when n is odd, follows from (18) by (5), since $h^2k \equiv k \pmod 8$; and for the same reason $(2/h^2k) = (2/k)$, by (15), and $(2/h^2k)(n/h^2k) = (2n/h^2k) = (2/k)(n/k) = (2n/k)$, showing that (19) holds when n is even.*

* Any relation regarding $(2n/k)$ holds good regarding $(2.4^rn/k)$, by (18). It might be stated that $(n/h)(n/k) = (n/hk)$, from which (19) follows; but this known theorem is not necessary to the algorithm. From the present point of view, it may be proved, for the case in which all the numbers are odd, by applying (5) to both sides of $(hk/n) = (h/n)(k/n)$, noting that $(-1)^{\frac{1}{4}(hk-1)} = (-1)^{\frac{1}{4}(h+k-2)}$, each side being $(-1)^{r+s}$ if $h = 2r + 1$ and $k = 2s + 1$; after which it may be shown similarly that $(2/h)(2/k) = (2/hk)$, producing $(2n/h)(2n/k) = (2n/hk)$. The relations (18) and (19) are known, but (19) cannot come into play when the law of reciprocity is employed in practice, and is therefore not well known.

15. Apart from the possibility of shortening the work by the throwing out of square factors when observed, the method now presented for the evaluation of the function (n/k), where k is odd and n any number prime to it, consists in reducing the larger of the two numbers n and k, by the help of certain rules, so as to make it the smaller, but still positive, always paying due attention to the sign of the function, then in reducing the other, and so on alternately until one or the other is reduced to 1, under the form $(1/k) = 1$ or $(n/1) = 1$. The sign of the result indicates the original value of (n/k) as $+1$ or -1. Under each rule of reduction the sign of the function is changed only in case some specified number happens to be oddly even, that is to say, to be of the form $4m + 2$, as, for example, 26.

16. *Rules of Reduction:*

	The sign of (n/k) is not changed when these numbers are reduced:	*Unless these numbers are oddly even:*
(20)	n to $n - xk$	[No exception]
(21)	n to $xk - n$	$k - 1$.
(22)	k to $k - 2xn$ or to $2xn - k$	n or $n - 1$ if x is odd
	[n even and x odd, the following:]	
(23)	k to $k - xn$ or to $xn - k$	both n and $k - x$.

These rules may also be formulated as follows:

(20) $$(n/k) = (n - xk/k),$$

(21) $$(n/k) = (-1)^{\frac{1}{2}(k-1)}(xk - n/k),$$

(22) $$(n/k) = (-1)^{\frac{1}{2}xn(n-1)}(n/ \pm k \mp 2xn),$$

(23) $$(n/k) = (-1)^{\frac{1}{2}n(k-x)}(n/ \pm k \mp xn) \text{ [n even, x odd]}.$$

Of the four rules, the first two, those relating to n, are known.

17. *Illustrations of the use of the four rules:*

Of (20): $(82/11) = (82 - 7.11/11) = (5/11)$.

Of (21): $(62/13) = (5.13 - 62/13) = (3/13)$.

$(82/11) = -(8.11 - 82/11) = -(6/11)$.

Of (22): $(19/79) = (19/79 - 4.19) = (19/3)$.

$(10/79) = (10/8.10 - 79) = (10/1)$.

$(38/79) = -(38/79 - 2.38) = -(38/3)$.

$(42/79) = -(42/2.42 - 79) = -(42/3)$.

Of (23):

$$(2/15) = (2/15 - 7.2) = (2/1).$$

$$(6/43) = (6/43 - 7.6) = (6/1).$$

$$(10/51) = -(10/51 - 5.10) = -(10/1).$$

$$(10/47) = -(10/5.10 - 47) = -(10/3).$$

For reducing n by (20) or (21), we have to divide n by k, using (20) if the least residue in absolute value is positive, (21) if it is negative. Thus for (82/11) the least residue is 5, so that (20) should be used, while for (62/13) the least residue is -3, indicating the use of (21). To reduce k, we must divide by $2n$ if n is odd, employing (22), without regard to the sign of the residue; but if n is even, we must divide k by n to obtain the least residue, likewise regardless of its sign, using (22) if the quotient is even or (23) if it is odd. Thus, to reduce 79 in (19/79), we must employ (22) because 19 is odd; to reduce 79 in (38/79), where n is even, the quotient is 2, and (22) must be used; but to reduce 51 in (10/51), 10 being even, we must employ (23) because the quotient, 5, is odd.

18. The rules of reduction, numbered (20) to (23), come directly from the elementary relations numbered (2), (1), (3) and (4) respectively. The last case requires attention. By repetition, if n be written for $2n$, the exponent of -1 is $\frac{1}{4}n[k-1+k-1-n+\cdots+k-1-(x-1)n]$, which is $\frac{1}{4}nx(k-1) - \frac{1}{8}n^2x(x-1)$. If n is divisible by 4, this $\equiv 0 \pmod 2$, and may not untruthfully be represented, as in (23), by $\frac{1}{4}n(k-x)$. If n is oddly even, both $\frac{1}{2}n$ and x are odd, and either may be suppressed as a factor, modulo 2. The expression thus becomes $\frac{1}{4}n(k-1) - \frac{1}{4}n(x-1) \equiv \frac{1}{4}n(k-x)$ as before. The second parts of (22) and (23) follow from the first, respectively, by reason of the theorem

$$(n/k) = (n/-k). \tag{24}$$

This follows, by (3), from

$$(n/2n-k) = (-1)^{\frac{1}{8}n(n-1)}(n/k), \tag{25}$$

which is proved as in paragraph 4.

19. For convenience, a negative value for k has been admitted in (24), although the definition $(n/k) = (-1)^{\mu(n,\,k)}$ relates only to positive values of n and k, and it must therefore be observed that (1) is compatible with negative values of n, by a fair extension, and (3) with negative values of k. It is in no case necessary, and is always undesirable, when dealing with actual numbers, to permit the appearance of the negative sign within the parenthesis which forms the function-symbol.

20. In addition to the four rules of reduction, we are also at liberty to avail ourselves, should it appear advantageous, of the rule of transposition known as the law of reciprocity (5). This rule may be stated thus:

	The sign of (k/k') is not changed when this transposition takes place:	*Unless these numbers are oddly even:*
(26)	(k/k') to (k'/k)	both $k-1$ and $k'-1$.

This recourse is of no value, as a rule, except when n is not much smaller than k. As an illustration of the exceptional case, the reduction $(6929/6931) = (6931/6929) = (2/6929) = (2/1) = 1$ is better than the reduction $(6929/6931) = (6929/6927) = (2/6927) = (2/1) = 1$; but it is only a little better, and four rules are remembered, or referred to, more easily than five. Apart from the exceptional case mentioned, it is possible to find, or to invent, other instances in which the law of reciprocity might be used to advantage, as may happen when its employment results in the appearance of a large square. For example, in one of LEGENDRE's numerical illustrations we find $(421/1459) = (1459/421) = (196/421) = (14^2/421) = 1$.* The difficulty in any such case consists in recognizing the rare opportunity beforehand.

21. In the function (n/k), k may or may not be a prime number. When it happens to be prime, a well known lemma of Gauss teaches that both sides of the congruence $n^c \cdot c! \equiv (n/k) \cdot c! \pmod{k}$, where $k = 2c+1$ (a congruence arising from the continued product of the terms $n, 2n \cdots, cn$), may be divided by $c!$, whence in this case $n^c \equiv (n/k) \pmod{k}$, which is 1 or -1 according as n is or is not a quadratic residue of k; because the residues, known to be roots of $n^c \equiv 1$, and non-residues, roots of $n^c \equiv -1$, are equal in number, and if the congruence $n \equiv x^2 \pmod{k}$ is possible, $(n/k) \equiv (x^2/k) = 1$. To learn whether n is a quadratic residue of k, a given prime, which is an important problem in the theory of numbers, it thus becomes necessary to determine whether $n^c \pmod{k}$, which when k is prime has the same value as one of our functions (n/k), is 1 or -1; and we may use for this purpose the algorithm which lowers alternately the values of n and k in the function (n/k), without inquiring whether the lower values of k are or are not prime.

IV. COMPARISON OF THE NEW ALGORITHM WITH PREVIOUS METHODS.

22. When $n > k$, the new method of reduction is the same as the old in requiring the employment of the two rules, numbered (20) and (21), which have

* Even in this first example of the sort which has presented itself, the regular method is equally lucky, giving $(421/1459) = (421/225) = (421/15^2) = 1$.

for their object the reduction of n. When $n < k$, n being odd, the new method substitutes the rule (22), which reduces k, for the rule (26), or law of reciprocity, which effects a transformation without reduction, the latter, however, being still available whenever use can be found for it. The sign is changed, by (22), n being odd, when $n-1$ is oddly even, and by (26), when both $n-1$ and $k-1$ are oddly even. As an illustration, the reduction $(13/29) = (13/3)$ is simpler than $(13/29) = (29/13) = (3/13) = (13/3)$, since one operation takes the place of three. In fact, when n is odd, the application of the rule (22) is exactly equivalent analytically to the application of (26) after (20) [or (21)], after (26), x being even in (20) or (21). In the following illustration of the old process, wherein k consists of the first few digits of π and n of those of the common logarithm of 2, only odd values appear:

$$\begin{aligned}(30103/314159) &= -(314159/30103) = -(13129/30103) = -(30103/13129)\\ &= -(3845/13129) = -(13129/3845) = -(2251/3845)\\ &= -(3845/2251) = (657/2251) = (2251/657) = (377/657)\\ &= (657/377) = (97/377) = (377/97) = (11/97) = (97/11)\\ &= (9/11) = 1.\end{aligned}$$

By using the third rule wherever $n < k$, instead of the law of reciprocity, this is shortened to

$$\begin{aligned}(30103/314159) &= -(30103/13129) = -(3845/13129) = -(3845/2251)\\ &= (657/2251) = (657/377) = (97/377) = (97/11) = (9/11) = 1.\end{aligned}$$

At each point where $n < k$, one operation serves instead of three.*

* The new third rule (22) is most valuable when applied to even values of n. Confined to odd values it is

(22a) $$(k/k') = (-1)^{\frac{1}{2}x(k-1)}(k/\pm[k'-2xk]),$$

being equivalent to a combination of the three known operations,

$$(k/k') = (-1)^{\frac{1}{4}(k-1)(k'-1)}(k'/k);$$

$$(k'/k) = (k'-2xk/k), \quad \text{or} \quad (k'/k) = (-1)^{\frac{1}{2}(k-1)}(2xk-k'/k);$$

and

$$(k'-2xk/k) = (-1)^{\frac{1}{4}(k-1)(k'-2xk-1)}(k/k'-2xk),$$

or

$$(2xk-k'/k) = (-1)^{\frac{1}{4}(k-1)(2xk-k'-1)}(k/2xk-k').$$

So simple a combination of these formulæ has probably not been looked for, yet that they might be consolidated in some way must have occurred to others, and may possibly have suggested the algorithms of EISENSTEIN and SYLVESTER. From another point of view, one branch of (22a), having + where (22a) has ±, has, for prime values of k and k', been used as a step, implicitly by BOUNIAKOWSKY, and explicitly by BUSCHE, in their demonstrations (as given by BAUMGART) of the law of reciprocity.

23. The law of reciprocity cannot be applied when n is even, and under the customary system it is therefore necessary, with $n < k$, to make n odd by reducing (n/k) to $(\frac{1}{2}n/k)$ by means of a fourth well-known rule, namely:

	The sign of (n/k) is not changed when this number is reduced:	*Unless one of these numbers is oddly even:*
(27)	n to $\frac{1}{2}n$	$\frac{1}{2}(k-1)$ or $\frac{1}{2}(k+1)$

Expressed as a formula, this fourth customary rule is

$$(n/k) = (2/k)(\tfrac{1}{2}n/k) = (-1)^{\frac{1}{8}(k^2-1)}(\tfrac{1}{2}n/k).$$

It is of course still available, like the old third rule, or law of reciprocity, in connection with the algorithm herein presented, if any unexpected use for it happens to appear, as for instance when $\frac{1}{2}n$ is a square, as in $n = 98$; but such cases must be rare. In lieu of it we have the fourth rule (23) of paragraph 16 if x is odd, otherwise the third rule (22). For example, by the customary method,

$$(10/103) = (2/103)(5/103) = (5/103) = (103/5) = (2/5) = -1;$$

but by the third rule, $(10/103) = -(10/3) = -(1/3) = -1$. Again,

$$(14/93) = (2/93)(7/93) = -(7/93) = -(93/7) = -(2/7) = -1;$$

but by the new fourth rule, $(14/93) = -(14/5) = -(1/5) = -1$. The object to be attained is the reduction of the larger number, k. By the new method, the object is effected instantly, leaving n the larger, ready for the next reduction. By the old method, n must be halved, by the rule here numbered (27); the law of reciprocity (26) must next be applied, producing $(k/\frac{1}{2}n)$; then k is reduced by (20) or (21); and finally the law of reciprocity must be applied again before the next reduction can be effected.

24. Having compared separately the steps to be taken when n is odd and when n is even, let us take a more extended example illustrating the use of all the rules [A] of the method heretofore in use and [B] of the method now set forth:

$$\begin{aligned}[A]\quad (1294/2477) &= (2/2477)(647/2477) = -(647/2477)\\ &= -(2477/647) = (111/647) = -(647/111) = (19/111)\\ &= -(111/19) = (3/19) = -(19/3) = -(1/3) = -1.\end{aligned}$$

[B] $(1294/2477) = -(1294/111) = (38/111) = (38/3) = -(1/3) = -1.$

Another case is that taken for illustration by TANNERY, (3988/887), which according to custom should run thus:*

$$(3988/887) = (440/887) = (4.110/887) = (110/887) = (2/887)(55/887)$$
$$= (55/887) = -(887/55) = -(7/55) = (55/7) = -(1/7) = -1.$$

Having now the means of depressing k directly, we can write

$$(3988/887) = (440/887) = (440/7) = -(1/7) = -1.$$

The old system depresses only n, while the new depresses n and k alternately, going, so to speak, on two legs instead of one, and so dispensing with the crutch known as the law of reciprocity.

25. Various substitutes have been proposed, without much success, for the method customarily employed for the evaluation of (n/k). BAUMGART describes such systems devised respectively by GAUSS, SYLVESTER, EISENSTEIN, LEBESGUE, GEGENBAUER and KRONECKER. Of these the simplest and best would seem decidedly to be that of EISENSTEIN, which has been chosen for reproduction in BACHMANN's recent text-book on the theory of numbers,† and which, indeed, by reason of its effective development and demonstration, based upon the law of reciprocity, is deserving of special remark. The practical working of EISENSTEIN's algorithm will alone be considered here.

26. EISENSTEIN's method provides for a succession of divisions with even quotients, so as to produce in each case the least possible positive or negative remainder. It may best be explained by numerical illustration. For present convenience, the successive divisions will be arranged in pairs, each pair occupying one line. The example taken is that chosen by BAUMGART, improved by an obvious correction and by observation of the detailed rules given by BACHMANN. When, as in this case, $n < k$, the first division is taken with the quotient 0. The evaluation of $(2933/3785)$ proceeds by the following steps:

* The evaluation given by TANNERY is erroneous.

† BACHMANN, in fact, does not illustrate, or even explain, the algorithm which I have described as customary, but replies to his own statement of the problem, "ist eine gegebene ungerade Primzahl q von einer andern ungeraden Primzahl p quadratischen Rest oder Nichtrest?" by "können wir dafür eine sehr einfache Regel angeben, welche EISENSTEIN aufgestellt hat."

Divd.		Qt.	Divr.		Rem.		Divd.		Qt.	Divr.		Rem.
2933	=	0 ·	3785	+	2933		3785	=	2 ·	2933	−	2081
2933	=	2 ·	2081	−	1229		2081	=	2 ·	1229	−	377
1229	=	4 ·	377	−	279	×	377	=	2 ·	279	−	181
279	=	2 ·	181	−	83	×	181	=	2 ·	83	+	15
× 83	=	6 ·	15	−	7	×	15	=	2 ·	7	+	1

Having continued the process till 1 is reached, we proceed to mark those cases in which the divisor is of the form $4r + 3$, namely, 279, 83, 15, and 7. As regards these marked cases, *for a first process*, count the number of remainders of the form $4r + 3$ and of the form $-(4r + 1)$. There are two, namely, -181 and 15. *For a second process*, count among the marked cases on the left side the number of negative remainders (namely, -7), and on the right side the number of remainders for which the quotient is not divisible by 4 (namely, -181, 15, and 1). By either process of counting the number is even, and this means that $(2933/3785) = 1$. The two processes are equally simple, or perhaps the second is less troublesome than the first, particularly in a long calculation.

27. What I have called the first process is that of Eisenstein. The second process, introduced merely for comparison, will be recognized as the present algorithm limited to odd values of n. The first, third, . . ., operations, those on the left side, are, for positive remainders, performed by the first rule (20) of paragraph 16, and for negative remainders by the second rule (21); while the alternate operations, those on the right side, are performed by the third rule (22). By the new algorithm, untrammeled, we should write $(2933/3785) = (2933/2081) = (852/2081) = (213/2081) = (213/49) = 1$.

28. A vital defect of Eisenstein's algorithm remains to be pointed out, apparently for the first time. It is a general system, yet there are cases to which its application is insufferably tedious. Take $(293/287)$. By the old algorithm this is solved at once thus, $(293/287) = (6/287) = (2/287)(3/287) = (3/287) = -(287/3) = (1/3) = 1$; and by the new thus, $(293/287) = (6/287) = (6/1) = 1$. But by the system of Eisenstein, accepted by one writer at least as an improvement on earlier methods, we should have

$$293 = 2.287 - 281 \qquad 287 = 2.281 - 275$$

$$281 = 2.275 - 269 \qquad 275 = 2.269 - 263$$

$$269 = 2.263 - 257 \qquad 263 = 2.257 - 251$$

and so on, requiring 42 more operations, with the subsequent counting of the cases in which the divisor is of the form $4r+3$ and the remainder of the same form or of the form $-(4r+1)$. The smaller the difference between n and k, other things being equal, the greater is the difficulty. Take $(293/97)$:

$$293 = 4.97 - 95 \qquad 97 = 2.95 - 93$$

$$95 = 2.93 - 91 \qquad 93 = 2.91 - 89$$

and so on, through 44 more operations, besides the subsequent counting. By either the old or the present method, this would be: $(293/97) = (2/97) = (2/1) = 1$.

MORRISTOWN, NEW JERSEY.

A DETERMINATION OF THE NUMBER OF REAL AND IMAGINARY ROOTS OF THE HYPERGEOMETRIC SERIES*

BY

EDWARD B. VAN VLECK

If the axis of x between 1 and ∞ is considered to be a cut, the hypergeometric series

$$F(a, \beta, \gamma, x) = 1 + \frac{a \cdot \beta}{1 \cdot \gamma} x + \frac{a(a+1)\beta(\beta+1)}{1 \cdot 2 \cdot \gamma(\gamma+1)} x^2 + \cdots,$$

with its analytic continuation, will define a function which is one-valued over the remainder of the plane of x. The number of roots of this function between 0 and 1 was ascertained first by STIELTJES† and HILBERT‡ for the special case in which $a = -n$, when the series reduces to a polynomial. Later the determination for the general case was effected by KLEIN§ in a memoir notable both for its results and for its method. The number of roots between 0 and $-\infty$ can be obtained from KLEIN'S results by means of the equation

$$F(a, \beta, \gamma, x) = (1-x)^{-a} F\left(a, \gamma - \beta, \gamma, \frac{x}{x-1}\right).$$

So far as I am aware, the number of imaginary roots has not been known, and is ascertained for the first time in the present paper.

For this purpose KLEIN'S geometrical method has been further developed. In the memoir above cited KLEIN made use of the conformal representation which is effected by the quotient of any two solutions of the hypergeometric differential equation. This quotient, as SCHWARZ showed, builds the positive half plane of x upon a triangle, bounded by arcs of circles, the sides of which

* Presented to the Society October 26, 1901. Received for publication December 3, 1901.

†Comptes Rendus, vol. 100 (1885), p. 620.

‡Crelle, vol. 103 (1887), p. 337.

§Mathematische Annalen, vol. 37 (1890), p. 573.

Other methods of finding the number have been given since by the following writers: HURWITZ, Mathematische Annalen, vol. 38, p. 452; GEGENBAUR, Wiener Berichte, vol. 100, p. 225, and Monatshefte für Mathematik und Physik, vol. 2, p. 124; PORTER, American Journal of Mathematics, vol. 20, p. 193, and Bulletin of the American Mathematical Society, vol. 6, p. 280. The simplicity of the form in which the results are obtained by HURWITZ is worthy of note.

correspond to the three segments into which the axis of x is divided by the singular points, $0, 1, \infty$, of the differential equation. KLEIN derives a formula for the number of times which any side returns upon or overlaps itself, and shows that either this number, or this number increased by 1, must be equal to the number of roots of any real solution of the differential equation within the corresponding segment of the axis. By taking the side which corresponds to the segment $(0, 1)$, the number of roots of the hypergeometric series between 0 and 1 is determined to within a unit. To decide, however, between the two values thus obtained, KLEIN abandons the triangle and settles the question by considering the sign of $F(\alpha, \beta, \gamma, x)$ when x approaches 1.

This departure from the fundamental principle of many of his investigations, —to wit, the determination of the properties of the integrals of a differential equation from the shape of the corresponding triangle—is, however, unnecessary. For, as will be shown here, the number of roots of certain particular integrals in each segment of the real axis can be ascertained directly from the triangle. These integrals correspond to the exponents of the singular points. Since $F(\alpha, \beta, \gamma, x)$ is such an integral, the number of its roots in each interval of the axis can be determined without any other aid than the triangle.

The completion of KLEIN's method leads immediately to the determination of the number of roots of the hypergeometric series in the imaginary domain. The theory can also be extended to any regular linear differential equation of the second order with real parameters (real singular points, exponents, etc.). If the analytic continuation of its solutions across the real axis from the positive into the negative half plane is forbidden, the fundamental integrals which correspond to the exponents of the singular points will define functions which are one-valued throughout the positive half plane. To find the number of roots of each function within the half plane, or in any of the segments into which the real axis is divided by the singular points, it is necessary only to construct the circular polygon into which the positive half plane is built by the quotient of any two solutions whatsoever of the differential equation.

One other important question is solved by means of the polygon. The differential equation in special cases may possess one or more integrals whose values are altered only by multiplicative constants for circuits around each of two or more singular points. The shape of the polygon reveals the existence or nonexistence of such integrals, and, when they exist, it indicates what integrals have this property.

I. THE GENERAL THEORY.

§1. *Notation and preliminary explanations.*

Let

$$p_0(x)\frac{d^2y}{dx^2} + p_1(x)\frac{dy}{dx} + p_2(x) = 0 \tag{1}$$

be any regular linear differential equation in which $p_0(x)$, $p_1(x)$, $p_2(x)$ are polynomials with real coefficients. We will suppose also that the roots of $p_0(x)$, which are the finite singular points of the differential equation, and the exponents of these points are real. The singular points will be denoted by $e_i (i = 1, 2, \cdots, n)$, the subscripts being so assigned as to indicate the order in which they succeed each other upon the axis of x. If the point at infinity is itself a singular point, we shall include it as the last of these points, e_n. Lastly, we shall denote the larger of the two exponents of e_i by λ_i', the smaller by λ_i''.

If the exponent difference $\lambda_i = \lambda_i' - \lambda_i''$ is not an integer, there are two integrals of the equation which in the vicinity of e_i have the form

$$P^{\lambda_i'} = (x - e_i)^{\lambda_i'}[1 + (x - e_i)P_1(x - e_i)],$$
$$P^{\lambda_i''} = (x - e_i)^{\lambda_i''}[1 + (x - e_i)P_2(x - e_i)],$$

P_1, P_2 being ordinary series in ascending powers of their arguments with real coefficients. We shall use the symbols $P^{\lambda_i'}$, $P^{\lambda_i''}$ to represent not merely the above expressions but also their analytic continuations over the positive half plane of x, inclusive of its boundary. Their continuation across the boundary is to be excluded. Upon this understanding $P^{\lambda_i'}$, $P^{\lambda_i''}$ are one-valued within the half plane, and each is a definite linear combination of $P^{\lambda_j'}$, $P^{\lambda_j''}$.

As is well known, the *conform* of the positive half plane which is obtained from the quotient of any two solutions of (1),

$$\eta = \frac{aP^{\lambda_i'} + bP^{\lambda_i''}}{cP^{\lambda_i'} + dP^{\lambda_i''}}, \tag{2}$$

is a polygon bounded by arcs of circles. The side which corresponds to the segment $e_i e_{i+1}$ will be denoted by $E_i E_{i+1}$. The angle at E_i is equal to $\lambda_i \pi$.

The term polygon must be interpreted from the point of view of the theory of functions. Not only may the point at infinity be contained within the polygon, but its surface may be composed of several leaves or partial leaves. If, for example, $\lambda_i > 2$, the surface will wind around E_i so as to overlap itself. It is possible also for a side to overlap, including one or more complete circumferences. We shall not find it necessary to enter into any further discussion of the form of the polygon except for the special case in which it is a triangle (§ 5). For any further information desired the reader is referred to KLEIN,* SCHÖNFLIES† and SCHILLING.‡

* *Lineare Differentialgleichungen* and *Hypergeometrische Function.*

† Mathematische Annalen, vols. 42 and 44.

‡ Ibid., vol. 44, p. 162.

§ 2. *On the connection between the roots of the fundamental integrals and the shape of the polygon.*

We shall now place

$$\eta = \frac{P^{\lambda_i'}}{P^{\lambda_i''}}, \tag{3}$$

so that E_i shall coincide with the origin. Since $P^{\lambda_i'}$, $P^{\lambda_i''}$ are real between e_i and e_{i+1}, the side E_iE_{i+1} is rectilinear and falls upon the real axis. The same two integrals will be real between e_{i-1} and e_i if multiplied by $e^{i\pi\lambda_i'}$ and $e^{i\pi\lambda_i''}$. Hence $E_{i-1}E_i$ is also rectilinear, making an angle $\lambda_i\pi$ with the axis. The second intersection of these two sides, produced if necessary, will be denoted by E_i'. In this case it lies at infinity.

Consider now the zeros of $P^{\lambda_i'}$ and $P^{\lambda_i''}$. It is a familiar fact that two independent integrals of (1) can vanish simultaneously only in the singular points. Such zeros need not be considered here. The remaining zeros of $P^{\lambda_i'}$ and $P^{\lambda_i''}$ give rise respectively to the zeros and infinities of η. The number of zeros of $P^{\lambda_i'}$ within the positive half plane of x is therefore equal to the number of times the polygon includes the origin of the η-plane in its interior, and the number of zeros in either of the segments $e_{i-1}e_i$ and e_ie_{i+1} is equal to the number of times the corresponding side passes through the η-origin. In general the remaining sides do not pass through the origin, and the real roots of $P^{\lambda_i'}$ are therefore usually included in the above segments. In special cases, however, some of the sides may pass through the origin, and every such passage of a side indicates the existence of a root in the segment corresponding to the side. The zeros of $P^{\lambda_i''}$ are indicated in like manner by the passage of the sides and interior of the polygon through the point at infinity.

Let now any other two solutions $\alpha P^{\lambda_i'} + \beta P^{\lambda_i''}$ and $\gamma P^{\lambda_i'} + \delta P^{\lambda_i''}$ be substituted for our two integrals. Obviously the polygon undergoes the transformation

$$\eta' = \frac{\alpha\eta + \beta}{\gamma\eta + \delta}. \tag{4}$$

The origin and point at infinity will be converted into the intersections of $E_{i-1}E_i$ and E_iE_{i+1} in the transformed polygon, but the number of times the surface of the polygon or any one of its sides passes over either intersection is in no wise altered by the transformation. We reach therefore the following result:

THEOREM. *If $\lambda_i' - \lambda_i''$ is not an integer, and if λ_i' denotes the larger of the two exponents of e_i, the number of zeros of $P^{\lambda_i'}$ within the positive half plane of x is equal to the number of times that the interior of the polygon corresponding to any two solutions of* (1) *passes over E_i. The number of its zeros in any segment of the axis between two successive singular points is equal to the number of times the corresponding side passes over E_i. The zeros*

of $P^{\lambda_i''}$ are indicated in like manner by the passage of the sides and interior of the polygon over E_i', the second intersection of the sides $E_{i-1}E_i$ and E_iE_{i+1}, produced if necessary.

When λ_i is a positive integer, the expression for the integral belonging to the larger exponent is the same as before, but, in general, the form of the other integral must be modified by the introduction of a logarithmic term so that it becomes

$$P^{\lambda_i''} = (x - e_i)^{\lambda_i''} P_2(x - e_i) + CP^{\lambda_i'} \log (x - e_i).$$

For the class of equations which we are considering, C and the coefficients of P_2 are real. The integrals have also the same form when the two exponents are equal. In this case necessarily $C \neq 0$, and the two integrals can therefore be distinguished by the presence or absence of a logarithmic term. We denote the non-logarithmic integral by $P^{\lambda_i'}$.

Suppose now that in all these cases we put η equal to the quotient (3). It is evident that when $x = e_i$, the quotient must vanish, and the vertex E_i will coincide again with the origin. From this it follows that

If λ_i is a positive integer or 0, the roots of $P^{\lambda_i'}$ will be indicated in the same manner as above.

We cannot, however, reach a similar conclusion concerning the other integral. For, though E_iE_{i+1} will coincide again with the real axis, $E_{i-1}E_i$ will be, in general, the arc of a circle tangent to the axis. The point at ∞ is therefore no longer an intersection of the two sides, and it is in no wise apparent what point in E_iE_{i+1} is to take the place of this point, when the polygon is transformed by (4).

§ 3. *On the coincidence of fundamental integrals belonging to two different singular points.*

The values of $P^{\lambda_i'}$, $P^{\lambda_i''}$ are altered only by a multiplicative constant when x describes a circuit around e_i. If, however, λ_i is an integer and $C \neq 0$, $P^{\lambda_i'}$ does not have this property. The case in which $C = 0$ is also an exceptional one, inasmuch as the two integrals are then multiplied by the same constant. Hence every solution of (1) must be modified in like manner. This is the only case in which any other independent solution shares with the two fundamental integrals the property under consideration. The occurrence of this exceptional case is shown at once by the polygon, for the two sides which meet in E_i at the angle $\lambda_i\pi$ are then arcs of a common circle, and only then. We may therefore dismiss from further consideration in this paragraph the singular points which correspond to such vertices, and confine our attention to the remainder.

We proceed to determine when there is a solution which is altered only by a multiplicative constant for a circuit around either of two singular points, e_i and e_j. One of the two fundamental integrals of e_i will then coincide, except for a

numerical factor, with a fundamental integral of e_j. Since this coincidence is a special property of the differential equation, it must, of course, manifest itself in some feature of the polygon which is unaltered by linear transformation.

Suppose first that the integrals which thus coincide are the two which belong to the larger exponents of e_i and e_j. We shall take $\eta = P^{\lambda_j'}/P^{\lambda_j''}$ so that E_j will be situated at the origin. Then, in consequence of the hypothesis just made, η must also have the form

$$\frac{aP^{\lambda_i'}}{cP^{\lambda_i'} + dP^{\lambda_i''}}. \tag{5}$$

But the latter expression vanishes for $x = e_i$. In other words, E_i coincides with the origin and hence with E_j.

Conversely, when these two vertices coincide, the two integrals differ only by a numerical factor. For if by a linear transformation of the η-plane the coincident vertices are brought to the origin, η will vanish both for $x = e_i$ and for $x = e_j$. It follows that $b = 0$ in (2), and η has accordingly the form (5). Since it has a similar form at e_j, we conclude that $P^{\lambda'}$ and $P^{\lambda_j'}$ coincide.

We will next suppose that $P^{\lambda_i'}$ and $P^{\lambda_j''}$ coincide. If η is taken as before and λ_j is not an integer, $E_{j-1}E_j$ and E_jE_{j+1} will not only meet at the origin but will be rectilinear. Accordingly E_j' lies at ∞. But in consequence of our hypothesis, η must also have the form

$$\frac{aP^{\lambda_i'} + bP^{\lambda_i''}}{cP^{\lambda_i'}}, \tag{6}$$

from which it follows that E_i lies at ∞ and coincides with E_j'. Conversely when these two points coincide (Fig. 1), the two integrals must coincide. For

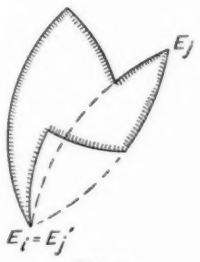


FIG. 1.

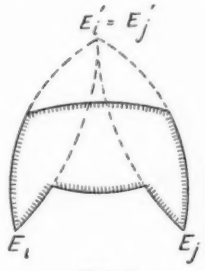


FIG. 2.

let E_j be brought to the origin by a linear transformation of η and at the same time let the two coincident points be removed to ∞. Then on the one hand η must take the form (6), while on the other hand it must be the quotient of the two fundamental integrals of e_j. It follows that $P^{\lambda_i'}$ and $P^{\lambda_j''}$ differ only by a numerical factor.

The coincidence of $P^{\lambda_i''}$ and $P^{\lambda_j''}$ may be discussed in similar fashion provided neither λ_i nor λ_j is an integer. The conclusion thus reached may be recapitulated as follows:

THEOREM.— *When the fundamental integrals $P^{\lambda_i'}$ and $P^{\lambda_j'}$, which belong to the larger exponents of e_i and e_j respectively, coincide except as to a numerical factor, this is revealed in the polygon by the coincidence of the vertices E_i and E_j. If E_i coincides with E_j', the second intersection $E_{j-1}E_j$ and E_jE_{j+1} (produced if necessary), and if λ_j is not an integer, $P^{\lambda_i'}$ and $P^{\lambda_j''}$ differ only by a numerical factor. Lastly, when neither λ_i nor λ_j is an integer, the coincidence of $P^{\lambda_i''}$ and $P^{\lambda_j''}$ is indicated by the coincidence of E_i', and E_j'* (Fig. 2).

An interesting application of this theorem may be made to the case in which three or more consecutive sides, or sides produced, pass through a common point. Let these sides be E_iE_{i+1}, $E_{i+1}E_{i+2}$, $\cdots$, $E_{i+r-1}E_{i+r}$. Then there is one integral which is modified only by a constant factor for circuits around any of the singular points e_{i+1}, e_{i+2}, $\cdots$, e_{i+r-1}. When all the sides pass through a common point, the polygon may be made rectilinear by removing the point to ∞. The differential equation then possesses an integral whose value is changed only by a constant factor for any circuit described in the x-plane. This equation and the corresponding polygon have been studied by KLEIN.

II. ON THE DISTRIBUTION OF THE ZEROS OF THE HYPERGEOMETRIC SERIES.

§ 4. *Introductory remarks.*

As is well known, the hypergeometric differential equation

$$x(x-1)\frac{d^2y}{dx^2} - (\gamma - (\alpha+\beta+1)x)\frac{dy}{dx} + \alpha\beta y = 0$$

has three singular points, $e_1 = 0$, $e_1 = 1$, and $e_3 = \infty$, with the exponent differences

$$\lambda_1 = |1-\gamma|, \quad \lambda_2 = |\gamma-\alpha-\beta|, \quad \lambda_3 = |\alpha-\beta|.$$

The two exponents of e_1 are 0 and $1-\gamma$, and the corresponding fundamental integrals are $F(\alpha, \beta, \gamma, x)$ and

$$F_1(x) = x^{1-\gamma}F(\alpha-\gamma+1, \beta-\gamma+1, 2-\gamma, x).$$

As we wish to consider here the functions which are obtained by continuing these two series analytically over the positive half plane of x—inclusive of its boundary—*the usual meaning of the symbols F and F_1 will be extended so as to include the two analytical continuations over this half plane.*

In accordance with § 2, we can find the number of roots of either integral in each of the segments into which the axis of x is divided by the singular points,

and also the number within the half plane, by constructing the triangle which corresponds to the differential equation. The theory fails only when λ_1, and hence γ, is an integer, and then only for the integral which belongs to the smaller exponent. Now when γ is a negative integer or zero, $F(a, \beta, \gamma, x)$, which is this integral, is devoid of meaning. If $\gamma = 1$, the two integrals coincide, and either is the non-logarithmic integral of e_1. Lastly, if $1 - \gamma$ is a negative integer, $F_1(a, \beta, \gamma, x)$ has no meaning. We conclude therefore that as long as either integral has a meaning, the distribution of its zeros can be obtained by the construction of the triangle.

By properly choosing the two solutions whose quotient is taken for the conformal representation, the vertices of the triangle may be made to take any assigned position. Its essential shape depends therefore only upon the magnitude of the angles. Since these are equal to $\lambda_i \pi (i = 1, 2, 3)$, our problem is to construct the triangle when the exponent differences are given.

§ 5. *Construction of the hypergeometric triangle.*

The construction of the triangle is usually somewhat complicated, but KLEIN* has shown how it can be constructed from a simpler or *reduced* triangle. By a reduced† triangle is to be understood one in which there is no angle greater than 2π and not more than one greater than π.

We shall first explain how the angles of the reduced triangle are to be obtained. For this purpose put

$$\lambda_1 = m_1 + \lambda_1', \quad \lambda_2 = m_2 + \lambda_2', \quad \lambda_3 = m_3 + \lambda_3',$$

in which m_i denotes the integral part of λ_i and λ_i' the fractional remainder. Two cases are to be distinguished. In the first, some one of the integers m_1, m_2, m_3 — call it m_i — is greater than the sum of the other two, m_j and m_k. We then make use of the *reduction:*

$$\begin{aligned} \lambda_i &= m_j + m_k + 2n + \lambda_i' + \epsilon_i, \\ \lambda_j &= m_j + \lambda_j', \\ \lambda_k &= m_k + \lambda_k', \end{aligned} \tag{7}$$

in which n is a non-negative integer and ϵ_i is equal to either 0 or 1. If then we set

$$\lambda_i'' = \lambda_i' + \epsilon_i, \quad \lambda_j'' = \lambda_j', \quad \lambda_k'' = \lambda_k', \tag{8}$$

the angles of the reduced triangle are $\lambda_i''\pi$, $\lambda_j''\pi$, $\lambda_k''\pi$.

* Mathematische Annalen, vol. 37. See also his *Hypergeometrische Function*, p. 404–424, where SCHILLING's definition of the reduced triangle is used.

† SCHILLING, loc. cit., p. 217.

In the second case each of the three integers m is equal to or less than the sum of the other two. In this case, if

$$M = m_1 + m_2 + m_3$$

is an even integer, place

$$(9) \qquad a_1 = \frac{m_2 + m_3 - m_1}{2}, \quad a_2 = \frac{m_3 + m_1 - m_2}{2}, \quad a_3 = \frac{m_1 + m_2 - m_3}{2}.$$

Then

$$(10) \qquad \lambda_1 = a_2 + a_3 + \lambda_1', \quad \lambda_2 = a_3 + a_1 + \lambda_2', \quad \lambda_3 = a_1 + a_2 + \lambda_3',$$

and to obtain the angles $\lambda_i''\pi$ of the reduced triangle we have merely to take

$$(11) \qquad \lambda_i'' = \lambda_i' \qquad (i = 1, 2, 3).$$

On the other hand, if M is an odd integer, we will set *

$$(12) \quad a_1 = \frac{m_2 + m_3 - m_1 - 1}{2}, \quad a_2 = \frac{m_3 + m_1 - m_2 \pm 1}{2}, \quad a_3 = \frac{m_1 + m_2 - m_3 \mp 1}{2},$$

so that a_1, a_2, a_3 will again be non-negative integers. Where the ambiguities in sign occur, the upper sign is to be selected unless $\lambda_2' \geqq \lambda_1' + \lambda_3'$, when, for a reason which will appear later, the lower sign should be taken. From (12) we get

$$(13) \quad \lambda_1 = a_2 + a_3 + \lambda_1', \quad \lambda_2 = a_3 + a_1 + \lambda_2' + \epsilon_2, \quad \lambda_3 = a_1 + a_2 + \lambda_2' + \epsilon_3,$$

where ϵ_2, ϵ_3 have the values 1 and 0 respectively, unless $\lambda_2' \geqq \lambda_1' + \lambda_3'$, when their values are to be interchanged. The angles of the reduced triangle are then specified by the equations:

$$(14) \qquad \lambda_1'' = \lambda_1', \quad \lambda_2'' = \lambda_2' + \epsilon_2, \quad \lambda_3 = \lambda_3' + \epsilon_3.$$

The various types of reduced triangles will be given later. After the proper one has been picked out, the construction of the triangle may be completed by the attachment of circles to the reduced triangle. The term *circle* is here to be understood in the general sense of the theory of functions. It may, according to circumstances, signify the portion of the plane within or without the bounding circumference, and in special cases the radius of the circle may be infinitely great so that the circle becomes a half plane.

Two modes of making the attachment have been given by KLEIN. By the first mode a circle is added *laterally* along a side, as in Fig. 3, where it is attached to $E_j E_k$. If two successive lateral attachments are made upon the same side, the one adds the portion of a plane exterior (interior) to the bounding circumference, and the other adds the portion interior (exterior) to the same circle. Hence the two together add an entire plane. Each lateral attachment on a side

* A very slight change is here made in the form of KLEIN's reduction.

$E_j E_k$ increases the angles E_j and E_k by π, and the triangle is bounded alternately by $E_j A E_k$ and the complementary arc $E_j B E_k$.

The second mode of adding a circle is known as *polar attachment*. A circle of the same radius as one of the sides $E_j E_k$ (Fig. 4) is taken and placed above

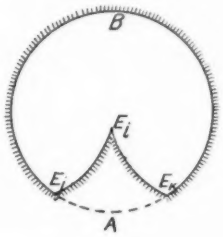


FIG. 3.

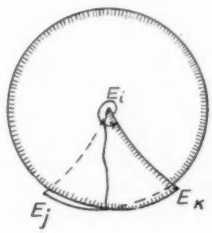


FIG. 4.

or below the triangle so that its circumference shall coincide, in part, with $E_j E_k$. A common cut is then made in the triangle and circle from the side to the opposite vertex, and the triangle and circle are then connected in the manner customary in the construction of a Riemann surface.* Each polar attachment increases a single angle by 2π and adds an entire circumference to the opposite side.

The reduced and completed triangles have, of course, the same vertices. An inspection of (7) and (8) shows that to complete the triangle when the first reduction is used, m_j and m_k lateral attachments must be made upon the sides $E_i E_j$ and $E_i E_k$, while n circles are to be hung to a cut from E_i to the opposite side. If the second reduction is employed, the construction is completed by the lateral attachment of a_1, a_2 and a_3 circles upon $E_2 E_3$, $E_3 E_1$ and $E_1 E_2$ respectively.

§ 6. *On the reduced triangle.*

All the various types of reduced triangles are shown in the accompanying plate.† The triangles are there divided into three sets of five each, which correspond to the three distinct positions which three intersecting circles may take relatively to one another. If the circles pass through a common point, this point may be removed to infinity by a linear transformation of η, and then the triangle becomes rectilinear as in the second section of the plate. *We shall pay*

* If one part of the triangle is placed above and one part below the circle before the pieces are connected, the completed figure will not intersect itself. See Fig. 4.

† Copied from KLEIN'S *Hypergeometrische Function*, p. 405. See also § 16 of the article by SCHILLING previously cited. Triangles 2 and 8 in the plate should be turned over in order that the interior may lie to the left of $E_i E_j$.

no special attention to the cases in which two of the circles are tangent to each other, since these are merely the limiting cases of those here considered.

For each position of the three circles there is a triangle in which the sum of the angles is equal to or less than the sum in any other triangle bounded by arcs of the same circles. This triangle is called the *minimal* triangle and is placed first in each of the three sections of the plate. If λ_0, μ_0, ν_0 denote the magnitudes of its three angles in terms of π, the triangle is distinguished from the remaining triangles by means of the relations

$$\lambda_0 + \mu_0 \leqq 1, \quad \mu_0 + \nu_0 \leqq 1, \quad \nu_0 + \lambda_0 \leqq 1,$$

and it will belong to section I, II or III according as

$$\begin{aligned} \lambda_0 + \mu_0 + \nu_0 &> 1 \text{ (triangle 1)}, \\ &= 1 \text{ (triangle 2)}, \\ &< 1 \text{ (triangle 3)}. \end{aligned} \tag{15}$$

The angles of the remaining reduced triangles are expressed in terms of λ_0, μ_0, ν_0.

The expressions for the angles given in the plate will enable us to decide which of the reduced triangles should be selected for given values of $\lambda_1, \lambda_2, \lambda_3$. It will not, however, be necessary in the subsequent work to distinguish between triangles 1 and 4, nor among 7, 10 and 13.

For convenience of treatment, we shall first divide the triangles into two groups, the first group comprising nos. 1–6, in which all the angles are acute, while the second group contains the remainder. An inspection of the reduction processes will show that if M is an even integer, $\lambda_i'' = \lambda_i' (i = 1, 2, 3)$, and consequently the angles of the reduced triangle will all be acute. On the other hand, if M is odd, one angle will be obtuse.

In the first group we have already distinguished the first three triangles from the others. No. 6 is characterized by a relation of the form

$$\lambda_j' + \lambda_k' - \lambda_i' = (1 - \mu_0) + (1 - \nu_0) - \lambda_0 > 1$$

or

$$\lambda_j' + \lambda_k' > 1 + \lambda_i', \tag{16}$$

while for no. 5 we have

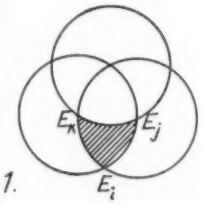
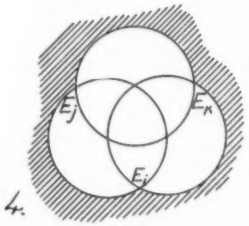
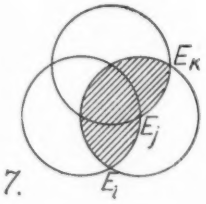
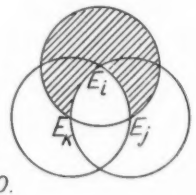
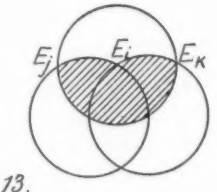
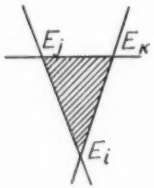
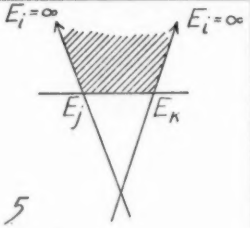
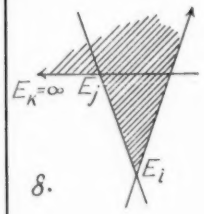
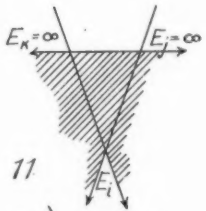
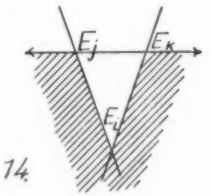
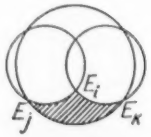
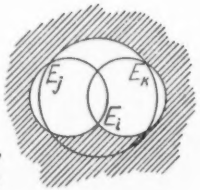
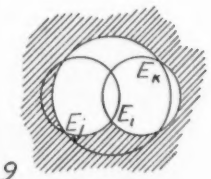
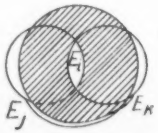
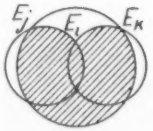
$$\lambda_j' + \lambda_k' = 1 + \lambda_i'. \tag{17}$$

In the second group of reduced triangles let the obtuse angle be denoted by $\lambda_r''\pi$. Then $\lambda_r' = 1 + \lambda_r'$, while for each of the two remaining angles $\lambda_i'' = \lambda_i'$. The expressions for the angles of no. 9 give

$$(1 + \lambda_j') + \lambda_i' - \lambda_k' = (1 + \mu_0) + \lambda_0 - (1 - \nu_0) < 1,$$

or

$$\lambda_k' > \lambda_i' + \lambda_j', \tag{18}$$

Section I	 1.	 4.	 7.	 10.	 13.
$\lambda''_i, \lambda''_j, \lambda''_\kappa =$	$\lambda_0, \mu_0, \nu_0.$	$\lambda_0, 1-\mu_0, 1-\nu_0$	$\lambda_0, 1+\mu_0, 1-\nu_0$	$2-\lambda_0, \mu_0, \nu_0.$	$2-\lambda_0, 1-\mu_0, 1-\nu_0.$
Section II	 2	 5	 8.	 11	 14.
$\lambda''_i, \lambda''_j, \lambda''_\kappa =$	λ_0, μ_0, ν_0	$\lambda_0, 1-\mu_0, 1-\nu_0$	$\lambda_0, 1+\mu_0, 1-\nu_0$	$2-\lambda_0, \mu_0, \nu_0$	$2-\lambda_0, 1-\mu_0, 1-\nu_0$
Section III	 3	 6	 9.	 12.	 15
$\lambda''_i\ \lambda''_j\ \lambda''_\kappa =$	$\lambda_0, \mu_0, \nu_0.$	$\lambda_0, 1-\mu_0, 1-\nu_0.$	$\lambda_0, 1+\mu_0, 1-\nu_0$	$2-\lambda_0, \mu_0, \nu_0$	$2-\lambda_0, 1-\mu_0, 1-\nu_0$

in which *one of the two subscripts on the right hand side refers to the obtuse angle*. Similarly for triangle 12 we obtain

$$\lambda_i' > \lambda_j' + \lambda_k', \tag{19}$$

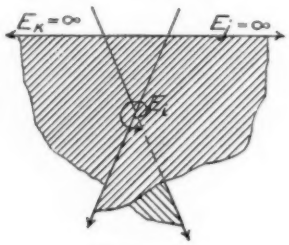
n which the subscript i refers to the obtuse angle. It can be easily verified that these relations hold for no other triangles of the second group. Triangle 15 is distinguished by means of the inequality

$$\lambda_1' + \lambda_2' + \lambda_3' > 2, \tag{20}$$

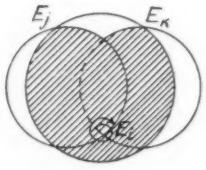
and the characteristic relations for nos. 8, 11, and 14 are obtained by merely replacing the sign $>$ by $=$ in the last three inequalities.

The possibility of making the attachments required by the reductions has yet to be considered. From a glance at the plate it is apparent that a circle can be attached laterally to any side of a reduced triangle with the exception of nos. 11 and 12. In each of these triangles lateral attachment upon E_jE_k, the side opposite to the obtuse angle, is impossible owing to the fact that the side returns upon itself. But it can be shown that such attachment is not required by the reductions. For, by (7) and (8), when the first reduction is used, the attachments upon the side opposite to an obtuse angle of the reduced triangle must be polar. On the other hand, when the second reduction is employed, it follows from (14) that E_2 is the vertex of the obtuse angle unless $\lambda_2' \equiv \lambda_1' + \lambda_3'$. The reduced triangle would then be of type 11 or 12. But in this exceptional case we so modified the form of the reduction as to make E_3 the vertex of the obtuse angle and thereby avoided the use of the two triangles. The requisite lateral attachments can therefore always be made.

Polar attachment is demanded only by the first reduction and the attachment is then made to a single side. If the reduced triangle has an obtuse angle, the side lies opposite to this angle. An inspection of triangles 7–15 and of 1–4



TRIANGLE 16.



TRIANGLE 17.

shows that in these triangles the attachment is always feasible. In nos. 5 and 6 polar attachment to E_jE_k is impossible, since in no. 5 the boundary of the half plane to be attached would pass through E_i, while in no. 6 the cut would

cross the boundary of the circle to be attached. When the form of the reduction leads to these exceptional cases, the construction of the triangle is to be effected as follows. Instead of making the first attachment triangles 16 and 17 are to be substituted. The angle E_i is thereby increased by 2π just as in polar attachment. The remaining $n-1$ polar attachments, as well as the lateral attachments required, may then be made in the usual manner. For convenience of statement we shall hereafter include these two triangles under the term reduced triangle.

§ 7. *On the distribution and number of roots of* $F(\alpha, \beta, \gamma, x)$ *when* $1-\gamma<0$.

All the needful preparation for the determination of the number of roots of $F(\alpha, \beta, \gamma, x)$ in each segment of the real axis and in the imaginary domain has now been made. When $1-\gamma<0$, $F(\alpha, \beta, \gamma, x)$ belongs to the larger exponent of e_1, and its roots are indicated by the passage of the sides and interior across E_1.

The number of real roots in the interval $(0, 1)$ is exactly equal to the number of times E_1E_2 crosses E_1 or overlaps itself. Now the only reduced triangles which contain an overlapping side are nos. 11, 12 and 16. But E_1E_2 will be identical with this side of no. 11 or no. 12 only when $m_3 > m_1 + m_2$ and then only if M is odd and $\lambda_3' \geqq \lambda_2' + \lambda_1'$. The corresponding conditions for no. 16 are that M should be even and $1 + \lambda_3' = \lambda_1' + \lambda_2'$. In every other case the overlapping of E_1E_2 is due solely to polar attachments upon this side. Each such attachment adds an entire circumference which covers E_1. Now the attachments upon E_1E_2 are polar only if $m_3 > m_1 + m_2$, and by (7) the number of such attachments (which we before denoted by n) is equal to the integral part of $(m_3 - m_2 - m_1)/2$. The first attachment, however, is not to be counted when triangle 16 or 17 is employed, that is to say, if $1 + \lambda_3' \geqq \lambda_1' + \lambda_2'$. The form of this condition suggests the introduction of the number*

$$E\left(\frac{\lambda_3 - \lambda_1 - \lambda_2 + 1}{2}\right)$$

in place of n, and this will be seen at once to agree exactly with the number of times E_1E_2 overlaps itself.†

Further attention should, perhaps, be called to triangles 11 and 16. In each of these triangles two vertices coincide. If the two are E_1 and E_2 the side E_1E_2 just closes, and there is a root of $F(\alpha, \beta, \gamma, x)$ in $e_2 = 1$. If this root is included in our enumeration, we reach the following result:

* By $E(q)$ is to be understood a number which is equal to the integral part of q if $q > 0$, and which is equal to 0 if $q \leqq 0$.

† We come thus to KLEIN's formula for the number of times any side overlaps itself.

If $1-\gamma<0$, *the number of roots of* $F(a, \beta, \gamma, x)$ *between* 0 *and* 1 *inclusive is* $E\{(\lambda_3-\lambda_1-\lambda_2+1)/2\}$.

From considerations of symmetry it follows immediately that *the number of roots between* 0 *and* $-\infty$ *must be* $E\{(\lambda_2-\lambda_1-\lambda_3+1)/2\}$.

To find the number of imaginary roots we must determine how often the surface of the triangle passes over E_1. Now the interior of a reduced triangle never crosses any of its vertices, and obviously it can only be made to do so by lateral attachment. If, in particular, it crosses E_1, the attachments must be made to the opposite side, E_2E_3.

Before taking up these attachments it will be convenient to dispose first of the case in which $m_1>m_2+m_3$. As the attachments upon E_2E_3 are then polar, we draw at once the following conclusion:

Case 1. *If* $1-\gamma<0$ *and* $m_1>m_2+m_3$, *the number of imaginary roots of* $F(a, \beta, \gamma, x)$ *within the positive half plane is equal to* 0.

We return now to the consideration of lateral attachments upon E_2E_3. When two consecutive lateral attachments are made upon any side of a reduced triangle, an entire plane is added which generally passes over the opposite vertex. The only exceptions are triangles 11 and 16 in which the vertices E_j and E_k coincide. It is therefore impossible to make the surface cross either vertex by lateral attachment. These two triangles can be obtained only if the first reduction is used and then only under the following conditions:

$$m_i>m_j+m_k \quad \begin{cases} M \text{ even}, & 1+\lambda_i'=\lambda_j'+\lambda_k', \\ M \text{ odd}, & \lambda_i' \quad =\lambda_j'+\lambda_k'. \end{cases}$$

But these are precisely the conditions which make $(\lambda_i'-\lambda_j'-\lambda_k'+1)/2$ an integer.

The effect of an even number of lateral attachments upon E_2E_3 has thus been ascertained. If the total number is odd, there remains one more attachment to be considered. Suppose first that M is even, and let a single circle be added laterally along a side of the reduced triangle. It will fail to cover the opposite vertex unless attached to E_jE_k in no. 6 or to a side ending in E_i in no. 17. As we are considering only attachments upon E_2E_3, the conditions for the occurrence of these exceptional cases must be

1) $$\lambda_2'+\lambda_3'>1+\lambda_1'$$

and

2) $$m_i>m_1+m_k, \quad \lambda_1'+\lambda_k'>1+\lambda_i'.$$

Suppose next that M is an odd integer. As by hypothesis $m_1\leqq m_2+m_3$, the vertex of the obtuse angle in the reduced triangle must be either E_2 or E_3. Hence we have only to consider the effect of a lateral attachment upon one of

the sides passing through the vertex of the obtuse angle. It will be found that the circle added will pass over the vertex opposite the side of attachment unless it is attached to triangles 11 and 12 or to $E_i E_j$ in 8 or 9. The conditions for the occurrence of these exceptional cases are

1) $$m_i > m_j + m_1, \quad \lambda_i' \equiv \lambda_j' + \lambda_1',$$

and

2) $$\lambda_1' \equiv \lambda_2' + \lambda_3'.$$

The effect of the attachments upon $E_2 E_3$ has now been completely determined. It remains only to ascertain their number and to apply our results. Three cases must be distinguished according to the number of attachments made.

Case 2: $m_2 > m_3 + m_1$. The form of reduction (7) shows that their number is

$$m_3 \equiv \frac{m_2 + m_3 - m_1 + 1}{2} - \frac{m_2 - m_3 - m_1 + 1}{2}.$$

Except in the special cases which have just been singled out the number of imaginary roots within the half plane will be $E(m_3/2)$ or $E\{(m_3+1)/2\}$ according as M is even or odd. The form of the conditions for the existence of the exceptional cases suggests, however, the introduction of the number

$$U = E\left(\frac{\lambda_2 + \lambda_3 - \lambda_1 + 1}{2}\right) - E\left(\frac{\lambda_2 - \lambda_3 - \lambda_1 + 1}{2}\right),$$

in terms of which our final result can be most simply expressed.

If $1 - \gamma < 0$ *and* $m_2 > m_1 + m_3$, *the number of imaginary roots of* $F(\alpha, \beta, \gamma, x)$ *within the positive half plane of* x *is equal to* $E(U/2)$ *unless* $\lambda_2 - \lambda_1 - \lambda_3$ *is an odd integer when the number is equal to* 0.

Case 3: $m_3 > m_1 + m_2$. The result is the same as in case 2 with the interchange of the subscripts 2 and 3.

Case 4: No one of the integers m_i greater than the sum of the other two. The number of lateral attachments upon $E_2 E_3$ is $E\{(m_2 + m_3 - m_1)/2\}$, and we obtain at once the following result:

If $1 - \gamma < 0$ *and each of the integers* m_1, m_2, m_3 *is equal to or less than the sum of the other two, the number of imaginary roots within the positive half plane is* $E(V/2)$, *where*

$$V = E\left(\frac{\lambda_2 + \lambda_3 - \lambda_1 + 1}{2}\right).$$

It is interesting to note how the changes in the number of imaginary roots take place when $\lambda_1, \lambda_2, \lambda_3$ are continuously varied. Since the roots of $F(\alpha, \beta, \gamma, x)$ are symmetrically situated with respect to the real axis and since

also a multiple root of any solution of the differential equation must coincide with a singular point, the change can conceivably take place in just two ways. Either a number of roots of $F(\alpha, \beta, \gamma, x)$ unite for an instant with a singular point and then separate and distribute themselves differently between the real and imaginary domains, or a root of a second branch of the function we are considering must cross the cut e_2e_3 and thus become a root of the branch $F(\alpha, \beta, \gamma, x)$.

Consider the first alternative. When $x = 0$, $F(\alpha, \beta, \gamma, x) = 1$. It is impossible therefore for roots of $F(\alpha, \beta, \gamma, x)$ to unite with e_1 so long as this symbol continues to have a meaning. We shall not consider here the changes which ensue when γ passes through a negative integral value. The union of the roots with e_2 or e_3 is shown by the coincidence of E_1 with E_2 and E_3 respectively. But, as we have seen, the number of imaginary roots is then 0. For an instant they are all absorbed into the singular point. It is possible also for two real roots to unite simultaneously with the same point, one being taken from each of the two segments which terminate in the point. Hence when the roots separate again, the number of imaginary roots in each half plane may be increased by a unit.

When the change takes place in the second manner and a root crosses the cut, E_2E_3 for the moment passes through E_1. The three sides of the reduced triangle then meet in a common point and it accordingly belongs to the second section of the plate. Now the only triangles of this section in which it is possible to make a side pass completely through the opposite vertex by lateral attachment are nos. 5 and 8. This happens when an odd number of attachments is made upon E_jE_k and E_iE_j respectively. If we impose the condition that E_2E_3 shall be this side, we obtain the following results:

If $1 - \gamma < 0$, *the number of real roots of* $F(\alpha, \beta, \gamma, x)$ *included between* 1 *and* ∞ *is equal to* 0 *unless* $(\lambda_2 + \lambda_3 - \lambda_1 + 1)/2$ *is a positive integer. Then if* m_3 *in case* 2, m_2 *in case* 3, *or* $E\{(m_2 + m_3 - m_1)/2\}$ *in case* 4 *is an odd integer, there will be a single root between* 1 *and* ∞, *and in no other case.*

§ 8. *On the number and distribution of the roots when* $1 - \gamma > 0$.

f $1 - \gamma > 0$, the roots of $F(\alpha, \beta, \gamma, x)$ are indicated by the passage of the sides and interior of the triangle across E_1', the second intersection of the sides E_1E_2 and E_1E_3. We will determine first the number of real roots, ascertaining for this purpose the number of times which E_1E_2 passes over E_1'.

Case 1: $m_1 > m_2 + m_3$.* When M is odd, the vertex of the obtuse angle is

* The same four cases are distinguished here as in the article by HURWITZ, but the number of roots is here expressed in terms of the exponent differences, while HURWITZ gives it in terms of α, β, γ. The change to the latter form is easily made.

E_1. The point E_1' is contained within E_1E_2 only if the triangle is of type 6, 9, 12 or 17, and one of the following sets of conditions must then hold:

(1) M is even and either $\lambda_1' + \lambda_3' > 1 + \lambda_2'$ or $\lambda_2' + \lambda_3' > 1 + \lambda_1'$;

(2) M is odd and either $\lambda_1' > \lambda_2' + \lambda_3'$ or $\lambda_2' > \lambda_1' + \lambda_3'$.

The lateral attachments upon E_1E_2 have no effect upon its position if their number be even, but if their number be odd, it must be replaced by the complementary arc. Now when one of the two complementary arcs contains E_1', the other will not. Exceptions arise only from the coincidence of E_1' and E_2, when both arcs terminate in E_1'. Such a coincidence occurs in figures 8 and 5 and then only if $\lambda_1' + \lambda_3' = \lambda_2'$ and $1 + \lambda_2' = \lambda_1' + \lambda_3'$ respectively. The total number of lateral attachments upon E_1E_2 is

$$m_2 \equiv m_1 - \frac{m_1 + m_3 - m_2 + 1}{2} - \frac{m_1 - m_2 - m_3 - 1}{2},$$

and the final result in each triangle depends upon whether this number is even or odd. Taking proper account of the exceptions noted, we obtain the following result:

If $1 - \gamma > 0$ *and* $m_1 > m_2 + m_3$, *the number of roots of* $F(\alpha, \beta, \gamma, x)$ *between* 0 *and* 1 *will be* 0 *or* 1 *according as*

$$X \equiv E(\lambda_1) - E\left(\frac{\lambda_1 + \lambda_3 - \lambda_2 + 1}{2}\right) - E\left(\frac{\lambda_1 - \lambda_2 - \lambda_3 + 1}{2}\right) \tag{21}$$

is even or odd, unless $(\lambda_1 + \lambda_3 - \lambda_2 + 1)/2$ *is an integer. In this special case there is a single root, in the interval, which coincides with* $x = 1$.

Case 2: $m_2 > m_1 + m_3$. If there is an obtuse angle, its vertex is E_2. Then E_1E_2 can not contain E_1'. As also the number of lateral attachments on this side is m_1, we conclude at once that

If $1 - \gamma > 0$ *and* $m_2 > m_1 + m_3$, *the number of roots between* 0 *and* 1 *will be either* 0 *or* 1 *according as* $E(\lambda_1)$ *is even or odd.*

Case 3: $m_3 > m_1 + m_2$. The attachments upon E_1E_2 are polar, and their number is equal to the integral part of

$$\frac{m_3 - m_2 - m_1}{2} = \frac{m_1 + m_3 - m_2}{2} - m_1.$$

Each adds a circumference containing E_1'. If M is odd, E_1E_2 lies opposite to the obtuse angle and contains E_1' unless $\lambda_2' > \lambda_1' + \lambda_3'$. If M is even, this point is included only if $\lambda_1' + \lambda_3' > 1 + \lambda_2'$ or $\lambda_1' + \lambda_2' \geqq 1 + \lambda_3'$. In the latter case the substitution of figure 17 for figure 6 takes the place of the first attachment. The final conclusion is as follows:

If $1-\gamma>0$ *and* $m_3>m_1+m_2$, *the number of roots of* $F(\alpha,\beta,\gamma,x)$ *in the interval* $(0,1)$ *is*

$$E\left(\frac{\lambda_1+\lambda_3-\lambda_2+1}{2}\right)-E(\lambda_1);$$

one of them coincides with $x=1$ *if* $\lambda_1+\lambda_3-\lambda_2$ *is an odd integer.*

We give without further discussion the result for

Case 4: *If* $1-\gamma>0$ *and if no one of the integers* m_1, m_2, m_3 *is greater than the sum of the other two, the number of roots in the interval* $(0,1)$ *will be either* 0 *or* 1 *according as*

$$E(\lambda_1)-E\left(\frac{\lambda_1+\lambda_3-\lambda_2+1}{2}\right)$$

is even or odd, unless $\lambda_1+\lambda_3-\lambda_2$ *is an odd integer. In the latter case there is a single root which coincides with* $x=1$.

We proceed next to determine the number of imaginary roots, observing for this purpose how often the interior of the triangle passes across E_1'. The only reduced triangle which can contain the point E_1' in its interior is no. 15, and E_1 must then be the vertex of the obtuse angle. This holds in case 1. Furthermore, this case is the only one in which the surface of the triangle can be made to include E_1' by polar attachment. On this account we shall postpone its consideration to the last.

In the remaining cases we have only to trace the effect of the lateral attachments. Each pair of consecutive attachments to a side adds an entire plane which necessarily contains E_1'. If the number of attachments is odd, there remains one more attachment to be taken account of. Suppose first that this is upon E_1E_2. Then if E_1' was originally contained within this side, it becomes an interior point in consequence of the attachment. Now we have already determined, in studying the number of real roots, under what conditions E_1' will be contained in E_1E_2 in the reduced triangle. The result applies with change of subscript to E_1E_3.

The effect of a single attachment upon E_2E_3 remains to be considered. If M is even, the circle attached will cover E_1' unless $\lambda_1'+\lambda_2'+\lambda_3'\leqq 1$ (triangle 3) or $\lambda_1'+\lambda_k'\equiv 1+\lambda_i'$. When M is odd, the side necessarily passes through the vertex of the obtuse angle (E_2 or E_3 being its vertex), and it will be seen that E_1' is made an interior point by the attachment only if $\lambda_1'+\lambda_2'+\lambda_3'>2$ (no. 15) or if, when E_j is the vertex of the obtuse angle, $\lambda_k'>\lambda_1'+\lambda_j'$ (no. 9).

We are now prepared for the consideration of case 2. If the number of attachments upon E_2E_3 is written in the form

$$\frac{m_1+m_2+m_3+1}{2}-\frac{m_1+m_2-m_3+1}{2},$$

we are led to express the final result as follows:

Case 2: *If* $1-\gamma>0$ *and* $m_2>m_1+m_3$, *the number of imaginary roots of* $F(\alpha,\beta,\gamma,x)$ *within the positive half plane of* x *is* $E(\lambda_1/2)+E(q/2)$ *in which*

$$q=E\left(\frac{\lambda_1+\lambda_2+\lambda_3+1}{2}\right)-E\left(\frac{\lambda_1+\lambda_2-\lambda_3+1}{2}\right)$$

unless $\lambda_1+\lambda_2-\lambda_3$ *is an odd integer, when the number is* $E(\lambda_1/2)$.

It is evident also that in

Case 3: $1-\gamma>0$, $m_3>m_1+m_2$. *The same result holds after the interchange of the subscripts* 2 *and* 3.

In case 4 there are three sets of lateral attachments to be taken account of. If M is odd, one of these is upon the side opposite to the obtuse angle. Now this side is E_1E_3 unless $\lambda_2'>\lambda_1'+\lambda_3'$ when the triangle is of type 9 and E_1E_3 coincides with E_iE_j. It follows that if M is odd, E_1' is contained in E_1E_3 unless $\lambda_2'>\lambda_1'+\lambda_3'$ or $\lambda_3'>\lambda_1'+\lambda_2'$. On the other hand, when M is even, this point is excluded unless $\lambda_1'+\lambda_2'>1+\lambda_3'$. These exceptions suggest that the introduction of

$$S=E(\lambda_1)-E\left(\frac{\lambda_1+\lambda_2-\lambda_3+1}{2}\right) \tag{22}$$

in place of a_2, the number of lateral attachments upon E_1E_3. For a corresponding reason we shall express the number of attachments upon E_2E_3 in the form:

$$a_1=E\left(\frac{m_1+m_2+m_3+1}{2}\right)-E\left(\frac{m_1-m_2+m_3+1}{2}\right)-E\left(\frac{m_1+m_2-m_3+1}{2}\right).$$

Finally, to simplify the result, we shall introduce analogous expressions in terms of the exponent differences. The simplest form for the result which I have been able to find is the following:

If $1-\gamma>0$ *and if no one of the integers* m_1, m_2, m_3 *is greater than the sum of the other two, the number of roots of* $F(\alpha,\beta,\gamma,x)$ *within the positive half plane is equal to*

$$E(W/2)+E(S/2)+E(T/2)+\epsilon',$$

in which S *is defined by equation* (22), T *is a like expression with the subscripts* 2 *and* 3 *interchanged, and*

$$W=E\left(\frac{\lambda_1+\lambda_2+\lambda_3+1}{2}\right)-E\left(\frac{\lambda_1-\lambda_2+\gamma_3+1}{2}\right)-E\left(\frac{\lambda_1+\lambda_2-\lambda_3+1}{2}\right),$$

while $\epsilon=1$ *or* 0, *the former value being taken unless* M *is even and simultaneously neither* $\lambda_1'+\lambda_2'>1+\lambda_3'$ *nor* $\lambda_1'+\lambda_3'>1+\lambda_2'$, *or unless* M *is odd and either* $\lambda_3'>\lambda_1'+\lambda_2'$ *or* $\lambda_2'>\lambda_1'+\lambda_3'$.

We return now to the case in which $m_1 > m_2 + m_3$. The interior of the reduced triangle will contain E_1' only if $\lambda_1' + \lambda_2' + \lambda_3' > 2$. Each circle added by polar attachment to E_2E_3 will cover E_1' if this point lies on the same side of E_2E_3 as does E_1, or, in other words, if the reduced triangle belongs to the third section of the plate. The total number of polar attachments is

$$E\{(m_1 - m_2 - m_3)/2\},$$

but it should be remembered that if $\lambda_2' + \lambda_3' > 1 + \lambda_1'$ and M is even, triangle 17 is to be substituted for no. 6, and this substitution takes the place of the first polar attachment. The effect of the lateral attachments depends upon the number X which we introduced in considering the number of real roots in case 1, but in certain cases this number should be replaced by $X + 2$. If we introduce a similar number to correspond to the side E_1E_3, we arrive at the following result:

Case 1: *If* $1 - \gamma > 0$ *and* $m_1 > m_2 + m_3$, *the number of imaginary roots of* $F(\alpha, \beta, \gamma, x)$ *within the positive half plane is equal, in general, to*

$$E(X/2) + E(Y/2),$$

in which X *is defined by equation* (22), *and* Y *is a similar expression with the subscripts* 2 *and* 3 *interchanged. If, however, any one of the following sets of conditions is fulfilled*

$$(23)\quad \begin{cases} 1)\quad m_1 + m_2 + m_3 \text{ even and } \lambda_1' + \lambda_2' + \lambda_3' < 1 \text{ or } \lambda_i' + \lambda_j' > 1 + \lambda_k' \\ \qquad\qquad\qquad\qquad\qquad\qquad\qquad\qquad (k = 1, 2, \text{ or } 3), \\ 2)\quad m_1 + m_2 + m_3 \text{ odd and } \lambda_1' + \lambda_2' + \lambda_3' > 2 \text{ or } \lambda_k' > \lambda_i' + \lambda_j'. \end{cases}$$

the number of such imaginary roots must be increased by

$$E\left(\frac{\lambda_1 - \lambda_2 - \lambda_3 + 1}{2}\right) + \epsilon,$$

in which ϵ *is* 0 *unless either* M *is even and* $\lambda - \lambda_i' - \lambda_j' + 1$ $(i = 2, 3)$, *or* M *is odd and either* $\lambda_1' > \lambda_2' + \lambda_3'$, *or* $\lambda_1' + \lambda_2' + \lambda_3' > 2$, *when its value is* 1.

One interesting remark may be made concerning the number of real roots between $e_2 = 1$ and $e_3 = \infty$. If in (23) the sign of inequality is replaced by the sign of equality, we shall have the conditions that the triangle shall belong to section II of the plate. The side E_2E_3 passes through E_1', and the number of real roots between 1 and ∞ is then usually $E\{(\lambda_1 - \lambda_2 - \lambda_3 + 1)/2\}$. This shows that the additional imaginary roots, noted just above, enter the half plane by crossing the cut simultaneously. This is the only case in which the number of roots between 1 and ∞ ever exceeds 1.

We leave to the reader all further consideration of the transitional cases which arise when the triangle belongs to the second section of the plate.

In conclusion, it may be pointed out that the number of roots of $F_1(\alpha, \beta, \gamma, x)$ can be obtained by interchanging the conditions $1-\gamma>0$ and $1-\gamma<0$. The number of imaginary roots in the entire plane is, of course, double the number in the half plane.

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THE SECOND VARIATION OF A DEFINITE INTEGRAL WHEN ONE END-POINT IS VARIABLE*

BY

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The method applied in the following paper to the discussion of the second variation in the case in which one end-point is movable on a fixed curve, is closely analogous to that of WEIERSTRASS † in his treatment of the problem for fixed end-points. The difference arises from the fact that in the present case terms outside of the integral sign must be taken into consideration. As a result of the discussion the analogue of JACOBI's criterion will be derived, defining apparently in a new way the critical point ‡ for the fixed curve along which the end-point varies. The relation between the critical and conjugate points is discussed in § 4.

§ 1. *The expression for the variation of the integral.*

Consider a fixed curve D,

$$x = f(u), \quad y = g(u),$$

and a fixed point $B\,(x_1, y_1)$. Let C be a curve,

$$x = \phi(t), \quad y = \psi(t),$$

cutting D at $A\,(u = u_0, t = t_0)$, passing through $B(t = t_1)$, and making the integral

$$I = \int_{t_0}^{t_1} F(x, y, x', y')\, dt$$

a minimum with respect to values of the integral taken along other curves joining D and B, and lying in a certain neighborhood of C. The following assumptions are made: §

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† *Lectures on the Calculus of Variations*, 1879.

‡ The same as KNESER's "Brennpunkt." See his *Variationsrechnung*, p. 89.

§ Literal subscripts will be used to denote differentiation, partial when several variables are involved. The zero-subscript or $[\ ]_0$ means that in the function designated $t = t_0$, $u = u_0$. Unaccented letters refer to D; while accented letters refer to C.

1) The functions discussed are regular at the points considered;

2) $$[x_u^2 + y_u^2]_0 \neq 0; \quad x'^2 + y'^2 \neq 0, \text{ for } t_0 \leqq t \leqq t_1;$$

3) F satisfies the usual homogeneity condition

(1) $$F(x, y, \kappa x', \kappa y') = \kappa F(x, y, x', y') \qquad (\kappa > 0);$$

4) $$F(x_0, y_0, x_0', y_0') \neq 0.$$

When the integral is taken along a curve,

(2) $$\bar{x} = \phi(t) + \xi(t), \quad \bar{y} = \psi(t) + \eta(t),$$

the first variation can be put into the well-known form: *

(3) $$\delta I = [F_{x'}\xi + F_{y'}\eta]_{t_0}^{t_1} + \int_{t_0}^{t_1} [G_1\xi + G_2\eta]\,dt,$$

where

$$G_1 = F_x - \frac{d}{dt}F_{x'}, \quad G_2 = F_y - \frac{d}{dt}F_{y'}.$$

According to WEIERSTRASS † the second variation can be expressed in the form:

(4) $$\delta^2 I = [R]_{t_0}^{t_1} + \int_{t_0}^{t_1} [F_1 w'^2 + F_2 w^2]\,dt,$$

where

$$R = L\xi^2 + 2M\xi\eta + N\eta^2, \quad w = y'\xi - x'\eta,$$

the functions F_1, L, M, N, F_2 being defined by the following equations:

$$F_1 = \frac{1}{y'^2}F_{x'x'} = -\frac{1}{x'y'}F_{x'y'} = \frac{1}{x'^2}F_{y'y'},$$

$$L = F_{xx'} - y'y''F_1, \quad N = F_{yy'} - x'x''F_1,$$

(5) $$M = F_{x'y} + x''y'F_1 = F_{xy'} + x'y''F_1,$$

$$F_2 = \frac{1}{y'^2}(F_{xx} - y''^2F_1 - L') = -\frac{1}{x'y'}(F_{xy} + x''y''F_1 - M')$$

$$= \frac{1}{x'^2}(F_{yy} - x''^2F_1 - N').$$

In the first place by considering variations of the curve which pass through the end-points A and B considered as fixed, the following two necessary conditions for a minimum are found:

* See KNESER, loc. cit., § 4. The arguments of F and its derivatives are always x, y, x', y'.

† WEIERSTRASS'S *Lectures*, 1879.

I. *C must be an extremal* satisfying $G_1 = 0$ and $G_2 = 0$;*

II. *F_1 must be $\geqq 0$ along the arc AB of the curve C.*†

In the second place consider variations which do not pass through A. In asmuch as only a necessary condition is desired, ξ and η can be chosen in a special manner. Let ξ_0, η_0, ξ, η be defined by the equations:

$$\xi_0 = f(u_0 + \sigma) - f(u_0) = [x_u]_0 \sigma + [x_{uu}]_0 \frac{\sigma^2}{2} + \cdots,$$

$$\eta_0 = g(u_0 + \sigma) - g(u_0) = [y_u]_0 \sigma + [y_{uu}]_0 \frac{\sigma^2}{2} + \cdots, \tag{6}$$

$$\xi = \phi_1 \xi_0 + \phi_2 \eta_0,$$

$$\eta = \psi_1 \xi_0 + \psi_2 \eta_0, \tag{7}$$

where ϕ_1, ϕ_2, ψ_1, ψ_2 are functions of t satisfying the relations:

$$\phi_1(t_0) = \psi_2(t_0) = 1, \quad \phi_2(t_0) = \psi_1(t_0) = 0,$$

$$\phi_1(t_1) = \phi_2(t_1) = \psi_1(t_1) = \psi_2(t_1) = 0.$$

A curve (2) constructed with ξ and η as in (7) will be said to *belong to the class C*. It is evident that each particular curve C cuts D when $t = t_0$, and passes through B when $t = t_1$.

For these special variations ΔI can be expressed as a power series in σ, say

$$\Delta I = S_1 \sigma + S_2 \frac{\sigma^2}{2} + \cdots. \tag{8}$$

S_1 and S_2 can be calculated from δI and $\delta^2 I$. From (3) and (6), since C passes through B,

$$\delta I = -[F_{x'} x_u + F_{y'} y_u]_0 \sigma - [F_{x'} x_{uu} + F_{y'} y_{uu}]_0 \frac{\sigma^2}{2} + \cdots, \tag{9}$$

and therefore from (4) and (9),

$$S_1 = -[F_{x'} x_u + F_{y'} y_u]_0,$$

$$S_2 = -[F_{x'} x_{uu} + F_{y'} y_{uu} + L x_u^2 + 2M x_u y_u + N y_u^2]_0 + \int_{t_0}^{t_1} [F_1 \bar{w}'^2 + F_2 \bar{w}^2]\, dt,$$

where $\bar{w}$ and $\bar{w}'$ are the coefficients of σ in w and its derivative.

From (8) it follows that a third necessary condition for the existence of a minimum is

III. $$S_1 = 0.$$

* E. g., see KNESER, loc. cit., § 8.

† WEIERSTRASS'S *Lectures*, 1879.

This is the well-known condition for transversality.* It follows also from (8) that *if a minimum exists, S_2 must be $\geqq 0$ for all curves of class C.* The further discussion of S_2 is the principal object of this paper.

§ 2. *A condition which prevents S_2 from becoming negative.*

Suppose now that C satisfies the conditions I and III, and (instead of II) the condition that F_1 is > 0 along the arc AB. Transform (4) by adding with Legendre,

$$0 = -[vw^2]_{t_0}^{t_1} + \int_{t_0}^{t_1} \frac{d(vw^2)}{dt}\, dt.$$

The integrand becomes a homogeneous quadratic expression in w and w'. If for $t_0 \leqq t \leqq t_1$, a regular function v exists satisfying the discriminant relation

$$(v) \qquad v^2 - F_1(F_2 + v') = 0,$$

then $\delta^2 I$ becomes

$$(10) \qquad \delta^2 I = [R - vw^2]_{t_0}^{t_1} + \int_{t_0}^{t_1} F_1 \left[w' + \frac{vw}{F_1} \right]^2 dt.$$

The integral of (v) is expressible in terms of the integral of a linear equation. For when $v = -F_1 U'/U$,

$$(U) \qquad v^2 - F_1(F_2 + v') = \frac{F_1}{U}(F_1 U'' + F_1' U' - F_2 U) = 0.$$

Then

$$(11) \qquad S_2 = -\left[F_{x'} x_{uu} + F_{y'} y_{uu} + L x_u^2 + 2M x_u y_u + N y_u^2 + F_1 \overline{w}^2 \frac{U'}{U} \right]_0 + \int_{t_0}^{t_1} F_1 \left[\frac{U\overline{w}' - U'\overline{w}}{U} \right]^2 dt.$$

Assume the general integral of the differential equations $G_1 = 0$ and $G_2 = 0$, which are of the second order, to be

$$x = \phi(t, \alpha, \beta), \qquad y = \psi(t, \alpha, \beta),$$

where α and β are arbitrary constants. Suppose that these equations represent C when $\alpha = \beta = 0$. Then two particular integrals of (U) are †

$$\vartheta_1 = \begin{vmatrix} \phi_t & \phi_\alpha \\ \psi_t & \psi_\alpha \end{vmatrix}, \quad \vartheta_2 = \begin{vmatrix} \phi_t & \phi_\beta \\ \psi_t & \psi_\beta \end{vmatrix},$$

* KNESER, loc. cit., § 10.

† E. g., see WEIERSTRASS's *Lectures*.

where $\phi_t = \phi_t(t, 0, 0)$, etc. Suppose ϑ_1 and ϑ_2 to be linearly independent. Then the general integral of (U) is

$$U = c_1\vartheta_1 + c_2\vartheta_2. \tag{12}$$

Since ϑ_1 and ϑ_2 are linearly independent they satisfy the equation*

$$\vartheta_1\vartheta_2' - \vartheta_2\vartheta_1' = \frac{c}{F_1} \qquad (c \neq 0). \tag{13}$$

A particular integral (12) *can now be selected so that in* S_2 *the term outside of the integral vanishes.* Put

$$\begin{aligned} P &= \left[\frac{F_{x'}x_{uu} + F_{y'}y_{uu}}{x_u^2 + y_u^2}\right]_0 + L_0\cos^2\delta + 2M_0\sin\delta\cos\delta + N_0\sin^2\delta, \\ Q &= \left[F_1\frac{(y'x_u - x'y_u)^2}{x_u^2 + y_u^2}\right]_0 = [F_1]_0(x_0'^2 + y_0'^2)\sin^2(\gamma - \delta), \end{aligned} \tag{14}$$

where γ and δ are the angles at A which C and D make with the x-axis. Then from (11),

$$S_2 = -\left[P + Q\frac{U_0'}{U_0}\right]\left[x_u^2 + y_u^2\right]_0 + \int_{t_0}^{t_1} F_1\left[\frac{Uw' - U'w}{U}\right]^2 dt.$$

KNESER has shown† that if $F \neq 0$ at A, and D cuts C transversally, then D cannot be tangent to C. Therefore Q is $\neq 0$. Since, furthermore, the equation (13) holds when $t = t_0$, c_1 and c_2 can be so determined that

$$P + Q\frac{U_0'}{U_0} = 0. \tag{15}$$

Two such values are

$$c_1 = P\vartheta_2(t_0) + Q\vartheta_2'(t_0), \quad -c_2 = P\vartheta_1(t_0) + Q\vartheta_1'(t_0).$$

If $\mathrm{H}(t, t_0)$ denotes the particular integral of (U) formed with these constants, then

$$\mathrm{H}(t, t_0) = P\Theta + Q\frac{\partial\Theta}{\partial t_0}, \tag{16}$$

where

$$\Theta(t, t_0) = \begin{vmatrix} \vartheta_1(t) & \vartheta_2(t) \\ \vartheta_1(t_0) & \vartheta_2(t_0) \end{vmatrix}.$$

The integral H is useful in forming a function v to satisfy condition (v), at least when B is near A. For from (13) and (16), when $t = t_0$,

$$\mathrm{H}_0 = Q\frac{c}{[F_1]_0} \neq 0.$$

* See CRAIG, *Linear Differential Equations*, vol. 1, p. 54.
† loc. cit., § 30.

These results lead to the following theorem:

If $\mathrm{H}(t, t_0) \neq 0$ *for* $t_0 \leqq t \leqq t_1$, *then for curves of class* C, S_2 *can be expressed in the form*

$$S_2 = \int_{t_0}^{t_1} F_1 \left[\frac{Uw' - U'w}{U} \right]^2 dt,$$

which cannot become negative.

§ 3. *The necessary condition.*

By following still more closely the method of Weierstrass it can now be shown that the condition $\mathrm{H}(t, t_0) \neq 0$ $(t_0 \leqq t < t_1)$ is necessary for the existence of a minimum. Suppose that this condition does not hold but that H has a zero t_0' between t_0 and t_1. Then, as will be proved, variations of class C can be found which make S_2 and ΔI negative.

Integrate by parts the first term in the integrand of (4). Then

$$\delta^2 I = [R + F_1 w w']_{t_0}^{t_1} - \int_{t_0}^{t_1} w[F_1 w'' + F_1' w' - F_2 w] dt.$$

Consider the equation,

$$(U_\epsilon) \qquad F_1 U'' + F_1' U' - (F_2 - \epsilon) U = 0,$$

where ϵ is a constant. From the theory of linear differential equations, an integral H_ϵ of the equation (U_ϵ) exists, depending upon ϵ for its value and having the following properties:

1) It is regular for $t_0 \leqq t \leqq t_1$;

2) $[\mathrm{H}_\epsilon]_0 = \mathrm{H}_0$, $[\mathrm{H}_\epsilon']_0 = \mathrm{H}_0'$;

3) If $\eta > 0$ is selected arbitrarily, $\delta > 0$ can be found such that $|\mathrm{H}_\epsilon - \mathrm{H}| < \eta$ for $t_0 \leqq t \leqq t_1$, if $|\epsilon| < \delta$.

H and H′ can not both be zero at t_0'. For otherwise, since the functions involved are regular and $F_1 \neq 0$, the expansion of the left member of (U) could not be identically zero. From 3) therefore, δ can be chosen so small that when $|\epsilon| < \delta$, H_ϵ also vanishes between t_0 and t_1, say at $t_{\epsilon 0}$.

Curves can now be chosen of class C, such that w satisfies the equation (U_ϵ). For example, let ξ and η be defined for $t_0 \leqq t \leqq t_{\epsilon 0}$ by the equations

$$(17) \qquad \begin{aligned} w = y'\xi - x'\eta &= (y_0'\xi_0 - x_0'\eta_0) \frac{\mathrm{H}_\epsilon}{\mathrm{H}_0}, \\ x'\xi + y'\eta &= (x_0'\xi_0 + y_0'\eta_0) \frac{\mathrm{H}_\epsilon}{\mathrm{H}_0}; \end{aligned}$$

and for $t_{\epsilon 0} \leqq t \leqq t_1$, let $\xi = \eta = 0$. Then

$$(18) \qquad \delta^2 I = [R - F_1 w w']_{t_0}^{t_1} + \epsilon \int_{t_0}^{t_1} w^2 \, dt.$$

From (9) and (18) by calculation as before, and since H satisfies (15), it follows that

$$S_2 = \epsilon \int_{t_0}^{t_{\epsilon 0}} w^2\, dt\,.$$

The function w can not be identically zero unless H_ϵ is so; and by 3) H_ϵ can not vanish identically if δ is taken small enough, since H does not. Hence for certain functions ξ, η as in (7), $S_2 \neq 0$ and can be made positive or negative by taking values of ϵ opposite in sign. From § 1 therefore the arc AB can not make I a minimum.

If now the point A' defined on C by t_0' is said to be the *critical point* for the curve D, a fourth necessary condition can be stated as follows:

IV. *If the extremal C, which passes through the fixed point B and cuts the fixed curve D transversally, is to make the integral*

$$I = \int_{t_0}^{t_1} F(x, y, x', y')\, dt$$

a minimum, then B must not lie beyond the critical point defined by D on C; or analytically,

$$\mathrm{H}(t, t_0) \neq 0 \quad \textit{for} \quad t_0 \leqq t < t_1.$$

§ 4. *Relation between the conjugate and critical points.*

The point conjugate to A is defined* by the zero t_0'' of $\Theta(t, t_0)$, which is nearest to t_0. The functions Θ and H are both integrals of (U) of the form (12), and are linearly independent since $\Theta_0 = 0$ and $\mathrm{H}_0 \neq 0$. By a theorem concerning linear differential equations of the second order† their zeros must separate each other, and $\mathrm{H} = 0$ has therefore one root between t_0 and t_0''.

The expression for H involves the curvature of D at A linearly. The curvature is

$$\frac{1}{r} = \frac{x_u y_{uu} - x_{uu} y_u}{[x_u^2 + y_u^2]^{\frac{3}{2}}}. \tag{19}$$

By differentiating (1) for κ it is found that

$$x' F_{x'} + y' F_{y'} = F.$$

* WEIERSTRASS'S *Lectures*, and KNESER, loc. cit., §§ 24, 31.

† M. BÔCHER, *An elementary proof of a theorem of Sturm*, Transactions of the American Mathematical Society, vol. 2 (1901), p. 150.

From this equation and III the values of $F_{x'}$ and $F_{y'}$ at A can be determined, and by substitution in (16) H becomes

$$\mathrm{H}(t, t_0) = \left(\frac{P_1}{r} + P_2\right)\Theta + Q\frac{\partial\Theta}{\partial t_0}, \tag{20}$$

where

$$P_1 = \frac{F_0}{\sqrt{x_0'^2 + y_0'^2}\sin(\gamma - \delta)}, \tag{21}$$

$$P_2 = L_0\cos^2\delta + 2M_0\sin\delta\cos\delta + N_0\sin^2\delta.$$

Suppose C and A fixed, and D changeable but always transversal to C at A. Then if the expression (20) for H is put equal to zero and solved for r, the resulting function of t will express the value which the radius of curvature of D at A must have in order that t may determine the critical point for D. By the use of (13), (14) and (21) the function and its derivative are found to be

$$r = \frac{-P_1\Theta}{P_2\Theta + Q\dfrac{\partial\Theta}{\partial t_0}},$$

$$\frac{dr}{dt} = -\frac{c^2\sqrt{x_0'^2 + y_0'^2}}{\left[P_2\Theta + Q\dfrac{\partial\Theta}{\partial t_0}\right]^2}\frac{F_0}{F_1}\sin(\gamma - \delta).$$

The denominator of r vanishes once between t_0 and t_0'' for the same reason that H does. From 4) of § 1 the derivative dr/dt is $\neq 0$ and has the sign of

$$\frac{F_0}{F_1}\sin(\gamma - \delta).$$

The radius of curvature (19) is positive when its direction is related to that of the curve D for increasing u as the $+y$-axis is to the $+x$-axis; otherwise it is negative.

From these results the following theorems can be stated if it is supposed that $F_0 > 0$:

1) *The critical point for a curve D which cuts the extremal C transversally at A, always lies between A and its conjugate A''.*

2) *The position of the critical point is determined by the curvature of D at A.**

3) *If the radius of curvature of D at A is supposed to vary continuously from 0 to ∞ on the same side of D as the arc AB, and from ∞ to 0 on the*

* See KNESER, loc. cit., p. 111.

opposite side, then the critical point moves continuously from A to A″ when there is a minimum, and from A″ to A when there is a maximum.

§ 5. *Relation between the preceding results and those of Kneser. Sufficient conditions.*

KNESER has derived a necessary condition which corresponds to IV. He shows that it is possible to find a set of extremals,*

$$x = \xi(t, a), \quad y = \eta(t, a), \tag{22}$$

each cutting D transversally when $t = t_0$, and giving C for $a = 0$. The curve D is then represented in the vicinity of C by the equations

$$x = \xi(t_0, a), \quad y = \eta(t_0, a),$$

where a is the parameter. The condition III of transversality requires that

$$[F_{x'}\xi_a + F_{y'}\eta_a]_0 = 0$$

for every a near zero, since each curve of the system (22) is transversal to D. This equation can be differentiated for a and the derivatives of F expressed in terms of F_1, L, M, N, from equations (5). From (14) and (20), P and Q depend only upon the curvature and direction of D, and are independent therefore of the parameter representation. It follows that for $a = 0$,

$$\frac{\partial}{\partial a}[F_{x'}\xi_a + F_{y'}\eta_a]_0 = \left[P + Q\frac{\Delta'(0, 0)}{\Delta(0, 0)}\right][\xi_a^2 + \eta_a^2]_0 = 0,$$

where

$$\Delta(t, a) = \begin{vmatrix} \xi_t & \xi_a \\ \eta_t & \eta_a \end{vmatrix}.$$

$\Delta(t, 0)$ must therefore satisfy (15). It can be proved, as for ϑ_1 and ϑ_2, that $\Delta(t, 0)$ is also an integral of (U). Since both H and $\Delta(t, 0)$ are integrals of (U) satisfying (15) they must be linearly dependent. That is,

$$\mathrm{H}(t, t_0) = C\Delta(t, 0), \quad C \neq 0.$$

The condition IV can therefore be restated in KNESER's form:

IV′. *If C as in* IV *is to make I a minimum, then a necessary condition is*

$$\Delta(t, 0) \neq 0 \text{ for } t_0 \leqq t < t_1.$$

KNESER proves this condition† by discussing the case in which B coincides

* KNESER, loc. cit., §30.

† loc. cit., §25.

with the critical point A'. He shows that unless the envelope has a singular point of a particular kind at A', there is no minimum, and so none when B lies beyond A'. The result is a stronger condition than IV′, namely,

$$\Delta(t, 0) \neq 0, \quad \text{for} \quad t_0 \leqq t \leqq t_1. \tag{23}$$

But his proof does not hold if the envelope has the exceptional form mentioned.

The method given in § 3 applies when B lies beyond A', and then includes KNESER's exceptional case. It cannot be used when B and A' coincide. For then it is not certain that the integral Π_ϵ can be made to vanish between t_0 and t_1, and since w must vanish for $t = t_1$, the functions ξ, η cannot be constructed as in (17).

If the conditions II and IV are amended to read:

Π_a. $F_1 > 0$ *for points* (x, y) *on* AB, *and for any* $(x', y') \neq (0, 0)$,

IV_a. $H(t, t_0) \neq 0$ *for* $t_0 \leqq t \leqq t_1$,

then $\Delta(t, 0)$ satisfies (23). According to KNESER a field can be constructed about AB, and the four conditions I, Π_a, III, IV_a are sufficient conditions for the arc AB to make the integral a minimum.

THE UNIVERSITY OF MINNESOTA,
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ON THE PROJECTIVE AXIOMS OF GEOMETRY*

BY

ELIAKIM HASTINGS MOORE

Introduction.

In the present paper I wish to consider the axioms called by HILBERT† (1899) *the axioms of connection and of order* (I 1–7, II 1–5 of HILBERT's list), and called by SCHUR‡ (1901) *the projective axioms* of geometry.

HILBERT states (l. c., pp. 3, 21) that his body of axioms consists of independent axioms, that is, that no one of the axioms is logically deducible from the remaining axioms. This statement, accepted by HÖLDER§ and SOMMER,‖ is not accepted by SCHUR,¶ who holds that HILBERT's axioms I 3, 4, 5 are consequences of the axioms I 1–2, 7, II 1–5. For reasons explained below, this last remark cannot be accepted as valid.

There are, however, redundancies** in HILBERT's axioms I–II, the two axioms

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† HILBERT, *Grundlagen der Geometrie, Festschrift zur Feier der Enthüllung des Gauss-Weber-Denkmals in Göttingen*, Leipzig, 1899.

‡ SCHUR, *Ueber die Grundlagen der Geometrie*, Mathematische Annalen, vol. 55 (1901), pp. 265–292.

§ HÖLDER, *Anschauung und Denken in der Geometrie*, Leipzig, 1900; cf. p. 35.

‖ SOMMER, *Hilbert's Foundations of Geometry*, Bulletin, vol. 6 (1900), p. 292.

¶ loc. cit., pp. 266, 267, 271.

** As to the independence of the axioms I 1–7, II 1–5 HILBERT (l. c., p. 21) refers to his Göttingen lectures of the winter semester 1898–99, as reported by DR. VON SCHAPER and manifolded. By the kindness of DR. BOSWORTH-FOCKE I have seen a copy of this report. Proofs are given that (1) I 2 is independent of I 1; (2) I 5 of I 1–4, 6–7; (3) I 6 of I 1–5, 7; (4) II 4 of I 1–7, II 1–3; (5) II 3 of I 1–7, II 1–2, 4; (6) II 5 of I 1–7, II 1–4.—In contravention of the remark of SCHUR, I prove below further that (7) I 3 is independent of I 1–2, 4–7, II 1–5; (8) I 4 of I 1–2, 6–7, II 1–5; (9) I 5 of I 1–3, 6–7, II 1–5.

Each of these proofs involves the exhibition of a particular geometry, in which, e. g., in case (9) the relations I 1–3, 6–7, II 1–5 hold while the relation I 5 does not hold. This exhibition is in terms of a science supposed to exist without self-contradictions, and the proposition of independence is contingent on the correctness of this supposition. As science of reference one prefers the usual analysis of real numbers or a science (e. g., the analysis of complex numbers; euclidean n-dimensional geometry) whose existence is deducible from that of analysis.

Such proofs were used by PEANO in investigations on the foundations of arithmetic and geometry (l. c., p. 62) and they play a brilliant part in the work of HILBERT.

I 4, II 4 being deducible from I 1–3, 5, 7, II 1–3, 5. I establish these two conclusions in § 5 of this paper.*

In § 1 I give a simplified arrangement of the projective axioms of n-dimensional geometry ($n \geqq 2$). I use as elements the point and the line and the segment of a line, the line and the segment being certain sets of points, and give convenient definitions of the plane or 2-space, and the k-spaces in general ($0 \leqq k \leqq n$); the properties of the k-spaces are developed in § 3.

It is convenient here briefly to characterize the systems of projective axioms of 3-dimensional geometry given by PASCH † (1882), PEANO ‡ (1889, 1894), INGRAMI § (1899) and HILBERT (1899).

PASCH undertook to make pure geometry in a strict sense a purely deductive (abstract) science based on certain assumed (abstract) notions subject to certain assumed relations. He introduced as basal notions for geometry the point, the linear-segment (*gerade Strecke*) and the planar-segment (*ebene Fläche*), in terms of which the line and the plane were defined. The linear-segment and the planar-segment are certain sets of points.

PEANO, following PASCH, retained the point and the linear-segment; but he defined the line and the plane by means of these elements alone. His system, as simplified by INGRAMI, is given by SCHUR (l. c., p. 267 ff.)

HILBERT has as basal notions the point, the line, the segment of a line (connected with the notion *between*) and the plane. The line and the plane are by the Hilbert axioms (I 1, 7_1; I 3, 7_2) connected with certain sets of points (lying on the line; the plane), in such a way that we may (and for simplicity we do) identify them with those sets of points. For convenience of reference I set down the Hilbert axioms I 1–7, II 5; the axioms II 1, 2, 3 are in effect the axioms 2, 5, 3 of § 1 of this paper, while axiom II 4 is quoted in the theorem of § 2.

I 1, 2, 7_1. *Two distinct points A, B determine in every case a line g; we set $AB = g$ or $BA = g$. Any two distinct points of a line determine this line; that is, if $AB = g$ and $AC = g$, and $B \neq C$, then $BC = g$. On every line there are at least two points.*

I 3, 4, 5, 7_2. *Three non-collinear points A, B, C determine in every case a plane π; we set $ABC = \pi$. Any three non-collinear points of a plane determine this plane. If two points A, B of a line g lie in a plane π, then every point of the line g lies in π. On every plane there are at least three non-collinear points.*

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*This paper has been prepared in connection with my current Chicago seminar-course on the foundations of geometry and analysis, and queries and remarks of members of this course, in particular, of Mr. O. VEBLEN, have been a source of much stimulus.

† PASCH, *Vorlesungen über neuere Geometrie*, Leipzig, 1882.

‡ PEANO, *I principii di Geometria*, Torino, 1889 (cited in the following paper), *Sui fondamenti della Geometria*, Rivista di Matematica, vol. 4 (1894), pp. 51–90.

§ INGRAMI, *Elementi di Geometria per le scuole secondarie superiori*, Bologna, 1899 (citation of SCHUR).

I 7_3, 6. *There exist at least* [*one plane and*] * *four non-coplanar points. If two planes* α, β *have a point* A *in common, then they have at least a second point* B *in common.*

II 5. *Let* A, B, C *be three non-collinear points and* g *a line lying in the plane* ABC *and containing no one of the points* A, B, C; *if the line* g *contains a point within the segment* AB, *then it always contains either a point within the segment* AC *or a point within the segment* BC.

Here every one of two (three) points which determine a line (plane) is defined as a point of or on or in or lying in the line (plane).—In interpretation we agree: (1) I 2 has the full meaning of its original statement, viz.: If $AB = g$ and $CD = g$, and $B \neq D$, then $BD = g$. (2) I 4 has the specification: If $A_1A_2A = \pi$, $B_1B_2B = \pi$ and $C_1C_2C = \pi$, and A, B, C are non-collinear, then $ABC = \pi$. (3) In I 3, if $ABC = \pi$, then $BAC = \pi$ and $ACB = \pi$. (4) The determinations in I 1, 3 are unique determinations.

Clearly the body of axioms of a system depends essentially upon the choice of the basal notions of the system. In this connection a remark is pertinent with respect to one's attitude concerning the foundations of geometry. I suppose that if geometry † is taken to be a natural science—the science or a science of the space in which or according to which we live—it would, as is contended by PASCH and PEANO, be undesirable to introduce the line as a basal notion. The linear-segment seems to be a more fundamental notion. But we may discriminate between that part of geometry which establishes a body of postulates based as directly as may be on spatial experience or intuition, and that part which consists in the organization of the science on the basis of the accepted body of axioms; and so we understand that it may in the development of the theory be convenient to replace the body of primary notions and relations by another body of notions and relations, less fundamental, but, with respect to the deductive geometry, more convenient. I suppose that with this thought HILBERT introduced the line and the plane as basal notions in his abstract geometry.—It is understood that greater generality would be obtained by introducing the axioms as valid, as one says, for a limited region of space. In this connection reference is made to KLEIN's introduction ‡ of the ideal elements of projective geometry without the use of the parallel axiom, and to the remarks of PASCH (pp. 4, 18, 126), of PEANO (p. 75) and of SCHUR (pp. 267, 274).

As has been stated, SCHUR has remarked that in the body of projective axioms of HILBERT the axioms I 3, 4, 5 are redundant. This criticism is, however,

* The bracketed addition to the original form of I 7_3 is a necessary addition; it is an implication of the remarks with which HILBERT introduces the axioms I.

† For $n = 3$.—In case $n > 3$ the geometry is perhaps essentially abstract.

‡ Mathematische Annalen, vol. 6, p. 134.—Cf. SCHUR, ibid., vol. 39, p. 113, and vol. 55, p. 274.

incorrect. For if in the ordinary euclidean 3-dimensional space we take the points, the lines, the segments of lines, and two intersecting spheres as the points, the lines, the segments of lines, and the (only) planes of a new geometry one has a geometry in which I 1–2, 6–7, II 1–5 are satisfied, while all the statements I 3, 4, 5 are invalid. Thus I 3, 4, 5 are not logical consequences of the remaining axioms I, II. Indeed, one may prove * that (1) I 3 is independent of I 1–2, 4–7, II 1–5; (2) I 5 of I 1–3, 6–7, II 1–5. These two remarks together with the fact (already stated) that I 4 is deducible from I 1–3, 5, 7, II 1–3, 5 furnish a satisfactory account of the rôles of the axioms I 3, 4, 5 in the Hilbert system I, II.

The remark of SCHUR was connected with the fact that, as PEANO pointed out, the plane is capable of explicit definition. Apparently SCHUR directly identified the Hilbert plane with the Peano plane. But in a particular Hilbert geometry the plane postulated by HILBERT is to be identified with the plane defined by PEANO certainly only by the mediation (to some extent) of the axioms I 3–7, II 5 by which alone the Hilbert plane is conditioned. Pursuing this thought I find (§ 5) that by replacing in the Hilbert system axioms I 3, 5, 7, II 5 by axioms considerably milder one is still able to make this identification and thus to prove the axioms in their original form, as used by HILBERT.

Evidently, with PEANO and SCHUR, we may better define the plane explicitly than thus, by the mediation of axioms, implicitly. The implicit definitions of the k-spaces in n-dimensional geometry (cf. HILBERT, loc. cit., p. 71) would be still more undesirable. The explicit definitions of § 1 seem to be simpler than those of § 4, the analogues of the Peano-Ingrami definitions.

To indicate briefly the arrangement of this paper:

In § 1 I formulate a body of seven projective axioms of n-dimensional geometry ($n \geqq 2$); these for $n = 3$ correspond with HILBERT's axioms I, II as follows:

$$(1; 2; 3; \underline{4}; 5; \underline{6}; \underline{7}) = (\text{I } 1, 2, 7_1; \text{II } 1; \text{II } 3; \underline{\text{II } 5}; \text{II } 2; \underline{\text{I } 7_2, 7_3}; \underline{\text{I } 6});$$

the axioms I 3, 4, 5 are omitted, the plane (and the k-spaces, in general) being explicitly defined; in the cases underscored the correspondence is only approximate; in all cases the effort has been to give clear and precise expression to the meaning intended to be conveyed.—Axiom 4, the triangle-transversal axiom, is funda-

* A geometry for case (1) is the ordinary euclidean 3-dimensional geometry, with the omission of one plane; one for case (2) is the ordinary euclidean 2-dimensional geometry with the modification that the general plane ABC is the set of points, O apart, of the lines AO, BO, CO, the point O being the center of the circle inscribed in the triangle ABC.

One has the feeling that, even on the assumption that the Hilbert plane ABC is defined explicitly as identical with the Peano plane ABC, the criticism of SCHUR remains incorrect, in view of the fact that the plane order axiom II 5 relates to a line supposed to lie in the plane and is accordingly in this respect milder than PEANO's two triangle axioms fundamental in the theory of the Peano plane. A geometry showing the correctness of this conjecture I have not found.

mental in the theory of the k-spaces; it replaces (its two corollaries 3, 4) the two triangle axioms introduced by PEANO as fundamental for the theory of the plane, which replaced PASCH's plane order axiom, retained by HILBERT as axiom II 5. —Axiom 7, the axiom of limitation, in agreement with SCHUR (loc. cit., p. 271), I think desirably replaces HILBERT's I 6.

HILBERT's axiom II 4 is in § 2 proved as a theorem depending on axioms 1–6, in particular on axiom 4, the triangle-transversal axiom.

In § 3 the properties of the k-spaces are developed on the basis of the definitions of § 1 and the axioms 1–6. Comparison with another type of definition is made in § 4.

In § 5 I prove that in HILBERT's system with the body of axioms I 1–3, 5, 7, II 1–3, 5, or with the milder body of axioms I 1, 2, 3′, 5′, 7_1, $7_2'$, $7_3'$, II 1–3, 5′ one may identify the Hilbert plane with the plane defined in § 1, and that the axioms 1–6 ($n = 3$) of § 1 hold. Then it follows (by §§ 2, 3, 4) that the Hilbert axioms I 4, II 4 are deducible from the Hilbert axioms I 1–3, 5, 7, II 1–3, 5.

§ 1. *The projective axioms of n-dimensional geometry.*

In an abstract deductive geometry (of space of n dimensions, $n \geqq 2$) we consider a set of things called *points*, having properties fully* determined by the body of postulates or axioms laid down as the basis of the geometry. The space under consideration, call it *the fundamental space*, is the set of all points under consideration.

The projective axioms involve certain sets of points, called *lines*, and certain sets of collinear points, called *segments* (of lines). These lines and segments receive definition only implicitly by the mediation of the axioms.

The projective axioms may be grouped as follows:

1–4: *Axioms of conditioned existence and of definition;*

5–6: *Axioms of absolute existence;*

7: *Axiom of limitation.*

The reader will understand that the designations attached to the various groups of axioms are intended to indicate somewhat clearly, although roughly, what seem to be the principal functions of the various axioms in the body of axioms.

AXIOM 1: THE LINE. *Two distinct points A, B determine uniquely a set of points, the line AB. The points A, B belong to or lie on the line AB. A line is a set of points thus determined by certain two and indeed by any† two distinct points belonging to it.*

* In so far as they pertain to the arbitrary geometry of the system of geometries in question.

† This statement may be replaced by these three: *The line AB contains and thus is identical with the line BA. The line AB contains the line AC, where C is any point not A of the line AB. The line AC contains the point B.*

AXIOM 2: THE SEGMENT OF A LINE. *Two distinct points A, B of a line determine uniquely a set of points distinct from A, B and lying on the line; this set of points is the segment AB, with extremities A, B. The segment AB contains and thus is identical with the segment BA. A segment is a set of points thus determined by two distinct points, its extremities.*

Def. 1. A point C of the segment AB is said *to lie between* or *to separate** the extremities A, B of the segment, in notation ACB or BCA.— The notation indicates the *order-relation* of the three points.

AXIOM 3: THREE COLLINEAR POINTS. *Of three distinct collinear points A, B, C one and only one lies on the segment determined by the other two: of the three order-relations, ABC, BCA, CAB one and but one† holds.*

AXIOM 4: THE TRIANGLE* WITH TRANSVERSAL LINE. *A line which cuts‡ one side of a triangle externally and another side internally cuts the third side internally.*

In other words (cf. fig. 1):

AXIOM 4. *If A, B, C are three non-collinear points and if the line $\frac{DF}{DE}$ cuts the lines $\frac{BC}{BC}$, $\frac{AB}{CA}$ in points $\frac{D}{D}$, $\frac{F}{E}$ respectively, where $\frac{BCD}{BCD}$ and $\frac{AFB}{CEA}$, then it cuts the line $\frac{CA}{AB}$ in a point $\frac{E}{F}$ such that $\frac{CEA}{AFB}$.*

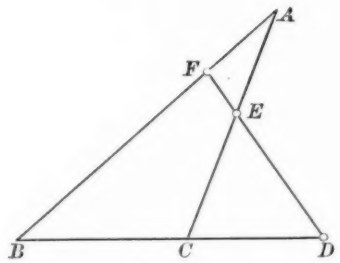

FIG. 1.

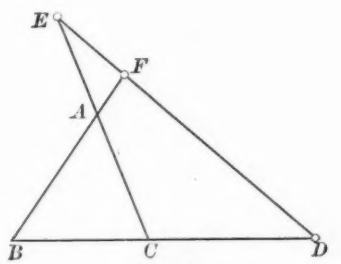

FIG. 2.

Notation. We denote the axioms into which axiom 4 has just been analyzed by the notations§

$$\begin{matrix} 4_1 \\ 4_2 \end{matrix} \quad \text{or} \quad \begin{matrix} 4_{11}, & 4_{12}. \\ 4_{21}, & 4_{22} \end{matrix}$$

*Similarly, a set Γ of points having a point C in common with the segment AB is said to separate A and B.

† Here, as Mr. VEBLEN has remarked, it is obviously sufficient to state that the relation ACB does not hold if the relation ABC does hold.

‡ For brevity I use, without explanation, a few geometric terms whose meaning in the abstract geometry in question is supposed to be sufficiently clear.

§ Similar self-explanatory partitional suffix-notations will be used elsewhere in this paper.

Remark. Axiom 4_{22} is a logical consequence* of axioms 4_{11}, 4_{12} and 4_{21} together with axioms 1–3.

Remark. The following seven theorems depend so immediately on axiom 4 by the mediation of the preceding axioms that I insert them here as corollaries of axiom 4.

Cor. 1. (In the notation of the second statement of axiom 4.)—*The three points D, E, F are distinct and such that DEF. The intermediate point E lies on the side adjacent to the point D lying on its side externally.*—One applies axiom 4_1 to the transversal AC of the triangle BDF.

Cor. 2. *If a line cuts two sides of a triangle externally and (cyclically) similarly, then it cuts the third side externally but dissimilarly, and the point of intersection with the third side lies between the points of intersection with the other two sides.*—That is (cf. fig. 2), in analogous notation: *If* $\frac{BCD}{BCD}$ *and* $\frac{CAE}{ABF}$, *then* $\frac{BAF}{ACE}$ *and* $\frac{DFE}{DEF}$.—One applies axiom 4_2 and cor. 1 to the transversal $\frac{BA}{AC}$ of the triangle $\frac{CDE}{BDF}$.—In this case the two statements are equivalent.

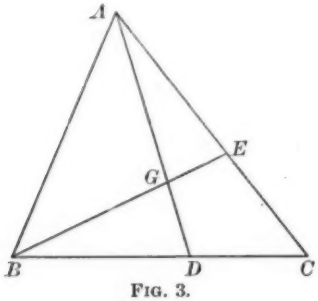

FIG. 3.

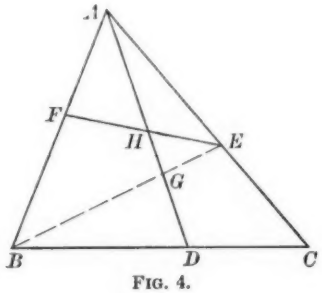

FIG. 4.

Cor. 3. *The two segments AD, BE from two vertices A, B of a triangle ABC to two points D, E of the respectively opposite segments BC, CA have in common a point G.*—One applies axiom 4_1 and cor. 1 to the transversal AD of the triangle BCE. Cf. fig. 3.

Cor. 4. *The point G being any point of the segment AD from the vertex A of a triangle ABC to a point D of the opposite segment BC, there exists a*

* By use of axioms 4_{21} and 3 we see that it is sufficient to prove that the relations ABF and BAF are impossible, and here we use besides axiom 4_1 itself its corollary 7 (of the text) which is independent of axiom 4_2.

The relation ABF is impossible. For by 4_1 the transversal FE of the triangle ABC would yield the relation BDC.

Further, the relation BAF is impossible. For, in conjunction with DEF, it would imply that the line AEC meets the triangle BDF in inner points of its three sides, which by cor. 7 is impossible; in conjunction with DFE, it would imply, by 4_1 for the triangle CED with the transversal BAF, the relation CAE; and similarly, in conjunction with EDF, it would lead to the relation ACE.

point E of the segment CA such that the segment BE contains the point G. —One applies axiom 4_2 and cor. 1 to the transversal BG of the triangle ACD.

Cor. 5. *In a triangle ABC the line AD, containing the vertex A and a point D of the opposite segment BC, and the segment EF, from a point E of the segment CA to a point F of the segment AB, have in common a point* H.*—One applies axiom 4_1 to the line AD as a transversal first of the triangle BCE and then of the triangle BEF. Cf. fig. 4.

Cor. 6. *In a triangle ABC the line AH, containing the vertex A and a point H of the segment EF, from a point E of the segment CA to a point F of the segment AB, contains a point D of the segment BC.*—One applies axiom 4_2 to the line AH as a transversal first of the triangle BEF and then of the triangle BCE.

Cor. 7. *No transversal meets the three sides of a triangle internally.*—If three points D, E, F on the sides BC, CA, AB of the triangle ABC were collinear, then, for instance, E would lie between D and F, and further, however, by cor. 5, the line AD would meet the segment EF in a point—in fact, the point D—lying between E and F.

Def. 2 ($k = 2$). Three distinct points A_1, A_2, B_0 are (in the order specified) *independent* if B_0 does not lie in the line A_1A_2.

Def. 3 ($k = 2$). Three independent points A_1, A_2, B_0 determine uniquely a *plane*, or 2-*space*, $A_1A_2B_0$. The plane $A_1A_2B_0$ is the aggregate set of all points A of the line A_1A_2, of all points C separated (cf. def. 1) from B_0 by the line A_1A_2, and of all points B separated from at least one of these points C by the line A_1A_2.

Def. 2 ($k = 3$). Four points A_1, A_2, A_3, B_0 are *independent* if A_1, A_2, A_3 are independent and B_0 does not lie in the plane $A_1A_2A_3$.

Def. 3 ($k = 3$). Four independent points A_1, A_2, A_3, B_0 determine uniquely a 3-*space* $A_1A_2A_3B_0$. The 3-space $A_1A_2A_3B_0$ is the aggregate set of all points A of the 2-space $A_1A_2A_3$, of all points C separated from B_0 by the 2-space $A_1A_2A_3$, and of all points B separated from at least one of these points C by the 2-space $A_1A_2A_3$.

Def. 2 ($k = k$). The $k + 1$ points $A_1, \cdots, A_k, B_0$ are *independent* if the first k points $A_1, \cdots, A_k$ are independent and the last point B_0 does not lie in the $(k - 1)$-space $A_1 \cdots A_k$.

Def. 3 ($k = k$).† A set of $k + 1$ independent points $A_1, \cdots, A_k, B_0$ de-

* It is to be noted that *on the basis of axioms* 1–5, *the point H lies not only on the segment EF but also on the segment AD.*—Denoting by G the point of intersection of the line AD and the segment BE, we have, by cor. 1, AGD and AHG, from which, by the theory of the order-relations of four collinear points (§2) we have indeed AHD.

† Denoting a point as a 0-space, we define similarly the notions: Two independent points A_1, B_0; the 1-space A_1B_0. Then within a fundamental space, which contains a 2-space, by the theory of §2, the 1-space A_1B_0 and the line A_1B_0 are identical. This remark makes general, for positive integral values of k, the notions of $k + 1$ independent points and of a k-space.—Evidently it does not involve an explicit definition of the notion *line*.

termines uniquely a k-space $A_1 \cdots A_k B_0$; the k-space $A_1 \cdots A_k B_0$ is the aggregate set of all points A of the $(k-1)$-space $A_1 \cdots A_k$, of all points C separated from B_0 by the $(k-1)$-space $A_1 \cdots A_k$, and of all points B separated from at least one of these points C by the $(k-1)$-space $A_1 \cdots A_k$.

AXIOM 5. *On the line AB determined by two distinct points A, B there exists a point C on the segment AB and there exists a point D distinct from A such that the segment AD contains B.*

AXIOM 6. *There exists in the fundamental space a set of $n+1$ independent points $A_1, \cdots, A_n, A_{n+1}$ (where n is any particular integer greater than 1); and thus there exists a set of $k+1$ independent points $A_1, \cdots, A_k, A_{k+1}$ for every integer $k (1 \leqq k \leqq n)$.*

Cor. *The fundamental space contains k-spaces $(1 \leqq k \leqq n)$.*

AXIOM 7. *There exists in the fundamental space* no $(n+1)$-space.*

Cor. *The fundamental space contains no set of $n+2$ independent points.*

In the system of axioms, axioms 6 and 7 may be condensed into the following

AXIOM. *The fundamental space is an n-space.*

§ 2. *The order-relations of four collinear points. Deduction from axioms* 1–4, 5_1, 6 $(n \geqq 2)$.

In the usual geometry one will admit the validity of the following four statements concerning the order-relations of four distinct collinear points A, B, C, D.

1. *If CAB and ABD, then CAD and CBD.*
2. *If CAB and ADB, then CAD and CDB.*
3. *If CAB and DAB, then either CDA or DCA; either CDB or DCB; if CDA, then CDB; and if CDB, then CDA.*
4. *If ACB and ADB, then either ACD or ADC; either CDB or DCB; if ACD, then CDB; and if CDB, then ACD.*

Concerning these four statements we establish the fundamental

THEOREM. *The statements* 1, 2, 3, 4 *are evidently equivalent to Hilbert's linear order axiom* II 4:

To any four distinct points of a line the notation A, B, C, D may always be assigned in such a way that ABC, ABD, ACD, BCD.

Furthermore, these statements are theorems of a geometry in which axioms 1–4, 5_1 *and* 6 *for $n \geqq 2$ hold.*

We prove, in the first place, by use of the axioms 2, 3 and 5_1, that the

* That in an n-space there exists no set of $n+2$ independent points and thus no $(n+1)$-space is a theorem. (Cf. § 3.)

statements are consequences* of the single statement 2_1, which, in modified notation, is:

2_1. *If* ABC *and* ACD, *then* BCD.

One observes readily that statements 3 and 4 follow from statements 1 and 2.

Now statement 2_2 follows from 2_1, for we have CAB and ADB, and if we had DBC, then by 2_1 from DBC and BAC we should have DBA in contradiction with ADB; and if we had BCD, then by 2_1 from DCB and CAB we should have DCA, and, at the same time, from ADB and DCB we should have ADC; these two conclusions are contradictory; hence we have, as stated, CDB.

Next the statements 1_1 and 1_2 are evidently equivalent, and we prove the truth of 1_1 on the assumption that 2_1 and consequently 2_2 are valid.

Changing the notation, we have ABC and BCD and are to prove ABD. If we had ADB, then by 2_1 from CBA and BDA we should have CBD, which is impossible. We must prove that DAB is impossible. Suppose we had DAB with ABC and BCD. If we had ACD, then by 2_2 from DCA and CBA we should have DBA, which is impossible. If we had DAC, then by 2_2 from DAC and ABC we should have DBC, which is impossible. Hence we should have CDA, and so the four points A, B, C, D would have the relations ABC, BCD, CDA, DAB; this set of four relations we denote by the notation $(ABCD)$ cyclic.

Accordingly we have at least proved that if ABC and BCD, then either ABD or $(ABCD)$ cyclic, and so likewise either DCA or $(DCBA)$ cyclic; that is, if ABC and BCD, then either ABD and ACD or else $(ABCD)$ cyclic.

We consider the case $(ABCD)$ cyclic. By axiom 5_1 there is in this case on the line $ABCD$ a point O different from A, B, C, D and such that DOA. By 2 from CDA, DOA, DAB we have CDO, COA, DOB, OAB.

Further, from BCD and CDO we have, in accordance with the alternative above proved, either BCO and BDO or else $(BCDO)$ cyclic. The former of these alternatives contradicts the result DOB obtained previously. Thus we have $(BCDO)$ cyclic, and from it DOB, OBC.

Similarly, from OAB and ABC we have either OAC and OBC or else $(OABC)$ cyclic. The latter of these alternatives is contradictory to the result OBC just obtained, while the former is contradictory to the result COA obtained earlier. Thus, by the intervention of the point O, we recognize that the case $(ABCD)$ cyclic is impossible. Hence 1 follows from 2_1.

Accordingly we have proved that the four statements 1–4 depend, by use of the axioms 2, 3 and 5_1, upon the single statement 2_1:

* Mr. VEBLEN has found that HILBERT's axiom II 4 is deducible from the two statements:— *If ABC and ACD, then ABD. If ABD and ACD, then ABC or CBD.*

If ABC and ACD, then BCD.

We are to prove further, by the use of axioms 1, 6 in addition to those previously mentioned, that this statement 2_1 depends upon axiom 4, the triangle-transversal axiom.

This will be proved in three parts:

(α) *The relations ABC, ACD, CDB cannot hold simultaneously.*

(β) *If ABC and ACD, then ABD.*

(γ) *The relations ABC, ACD, DBC cannot hold simultaneously.*

Here (β) depends upon (α) and is a lemma for (γ), while (α) and (γ) together yield the theorem in question.

In proof of (α): We take a point E, not on the line $ABCD$; we draw the distinct lines EC, EB; and on the segment EC we select a point F. Then, in accordance with axiom 4 and its first corollary as applied to the transversal AF of the triangle EBC, in view of the relations ABC and EFC, we see that the line AF cuts the segment EB in a point G for which EGB and AGF. In the triangle EBC the segment FD cutting the two segments EC, BC (in view of the relation CDB) is itself cut by the line CG, in accordance with cor. 5 of axiom 4, in a point H for which FHD. Then (in view of the relation ACD) the line CGH cuts the three sides of the triangle ADF in three inner points. But this, by cor. 7 of axiom 4, is impossible. Accordingly statement (α) is proved.

In proof of (β): The case ABC, ACD, BAD (that is, DCA, DAB, ABC) is excluded by (α), and the case ABC, ACD, ADB leads by (α) to this impossibility that no one of the three points B, C, D is between the other two. Accordingly statement (β) is proved.

In proof of (γ): We take a point E, not on the line $ABCD$; we draw the line EC; and on the segment EC we select a point F. Then, in view of EFC and ACD, the transversal AF cuts the side ED of the triangle ECD in a point G, where EGD and AFG.—Further, in view of ABC and AFG, the transversal GB of the triangle ACF cuts the segment CF in a point H; for this point H, in view of CHF and CFE, we have by (β) the relation CHE. Then the transversal BGH, in view of DBC, DGE and CHE, meets the three sides of the triangle DCE internally, and this is impossible.

§ 3. *Properties of the k-spaces of the fundamental space.*

We suppose that axioms 1–6 with $n \geqq 2$ hold for the fundamental space under consideration in § 3. By the first statement of the following theorem it appears that a k-space of the fundamental space is a fundamental k-space in the sense of § 1, the axioms 1–7 holding with k as n; those points, lines, segments of the fundamental space, which lie in the k-space, being directly identified with the points, lines, segments of the fundamental k-space.

THEOREM.—*In a fundamental space for which the axioms* **1–6** *with* $n \geqq 2$ *are valid, the sets of independent points and the k-spaces have the following properties:*

1. *Any two distinct points of a k-space determine a line or* **1***-space lying entirely in it.*
2. *A k-space is determined by any set of* $k+1$ *independent points lying in it.*
3. *A k-space contains no set of* $k+2$ *independent points.*
4. *Any* $k'+1$ *independent points of a k-space* $(0 \leqq k' \leqq k)$ *determine a* k'*-space lying entirely in it.*
5. *If* $k+1$ *points are independent when taken in a certain order, they are independent when taken in any order.*
6. *A* $(k-1)$*-space lying in a k-space separates the points of the k-space which do not lie in the* $(k-1)$*-space into two sets or parts in such a way that two points of* $\begin{matrix}\text{the same part are not}\\ \text{opposite parts are}\end{matrix}$ *separated by the* $(k-1)$*-space.*
7. *In a k-space, a p-space and a q-space* $(p+q \geqq k)$ *having a common point have in common precisely an r-space, where r is a definite integer such that* $r \geqq p+q-k$ *and (by* 3*)* $0 \leqq r \leqq p$, $0 \leqq r \leqq q$.

In view of the theorem of § 2 this theorem is true for the case $k=1$, a 0-space being a point. Its truth for the general k may be proved by the induction from k to $k+1$. The general step of induction will however be clearer if the proof for the case $k=2$ is given by itself.

For the case $k=2$ we consider three independent points A_1, A_2, B_0. We denote by a the line A_1A_2; by $\{A\}$ the set of points A of the line a; by $\{C\}$ the set of points C separated from B_0 by a; with respect to a particular point C, by $\{B_C\}$ the set of points B_C separated from the point C by a; by $\{\{B_C\}\}$ the aggregate of all such sets $\{B_C\}$. In these notations the plane $\pi = A_1A_2B_0$ is by def. 2 of § 1 a certain set $\pi = \{P\}$ of points P, viz:

$$\{P\} = \{A\} + \{C\} + \{\{B_C\}\}.$$

The proof of the statements of the theorem may easily be made to depend on the following three remarks:

The plane $\pi = A_1A_2B_0 = aB_0$ contains all the lines of the type AB_0.

The two planes $\pi = A_1A_2B_0$ and $\pi' = A_1A_2B_0'$, where B_0' is any point P of π not in a, are identical.

The two planes $\pi = A_1A_2B_0$ and $\pi' = A_1B_0A_2$ are identical.

In proof of the first remark the theorem of § 2 is at once applicable.

In proof of the second remark we have in mind axioms 4 and 5 and the theorem of § 2. In the plane $\pi = A_1A_2B_0 = aB_0$, the line a does not separate two points C. Further the sets $\{B_{C_1}\}$, $\{B_{C_2}\}$ connected with two distinct points C_1, C_2 of the set $\{C\}$ are identical; this is true if the line C_1C_2 contains a

point A, and likewise, if the line C_1C_2 contains the point B_0, and further if the two lines B_0C_1, B_0C_2 are distinct, in view of the fact that on the line B_0C_2 there is a point C_3 of the set $\{C\}$ such that the line C_1C_3 contains a point A. Thus the various sets $\{B_C\}$ are identical with one another; we set $\{B_C\} = \{B\}$ and have

$$\{P\} = \{A\} + \{C\} + \{B\} = \{A\} + \{B\} + \{C\}.$$

The sets $\{B\}$, $\{C\}$ have no points in common, since two points C_1, C_2 or B_1, B_2 are not separated by a, while two points B, C are separated by a. The line a separates the plane π into the two parts $\{B\}$, $\{C\}$. It is now easy to see that the plane $\pi = A_1A_2B_0$ is identical with the plane $\pi' = A_1A_2B_0'$, if B_0' is any particular point C, and so, by repetition of the argument, also if B_0' is any particular point B. Accordingly the second remark is proved.

From the second remark a point P of the plane $\pi = A_1A_2B_0 = aB_0$ is a point A of the line $a = A_1A_2$, or a point A' of line $a' = A_1B_0$, or a point P such that for certain two distinct points A_3 of a, A_3' of a' the relation $A_3'A_3P$ holds; in the last case selecting a point A_4 of a such that $A_3A_1A_4$, we find a point A_4' of a' such that $A_4A_4'P$. Accordingly the planes $A_1A_2B_0$ and $A_1B_0A_2$ are identical.

We proceed to prove the truth of the theorem in question for the case $k = m + 1$ on the assumption of its validity for the case $k = m$.

We consider the $(m+1)$-space $\Sigma = A_1A_2\cdots A_mA_{m+1}B_0 = aB_0$. One proves as before (but with the use of statement 1 of the theorem for $k = m$) that, in notations analogous to those previously used,

$$\Sigma = \{S\} = \{A\} + \{B\} + \{C\},$$

and that the $(m+1)$-space $\Sigma = aB_0$ is identical with the $(m+1)$-space $\Sigma' = aB_0'$, where B_0' is any point of Σ not in a.

Further, one sees that the $(m+1)$-space Σ contains the m-space a, that all lines joining points S to points A lie in Σ, and that the m-space a divides the space Σ into two parts as specified in statement 6.

By statement 2 for the m-spaces, the space Σ may then be obtained from any $m+1$ independent points of a together with any point S of Σ not in a.

Statements 1 to 6 will follow easily from the remarks already made, when we prove that the two* $(m+1)$-spaces,

$$\Sigma = A_1\cdots A_mA_{m+1}B_0, \quad \Sigma' = A_1\cdots A_mB_0A_{m+1}$$

are identical, and in proof of this last statement it is convenient to modify the notations. The $(m-1)$-space $A_1\cdots A_m$ is denoted by $d = D_1\cdots D_m$; the

* One uses statement 3 for $k = m$.

points A_{m+1}, B_0 are denoted by A_0, A_0'; and the lines $D_m A_0$, $D_m A_0'$ are denoted by l, l'. We have then

$$a = dA_0 = dl, \quad a' = dA_0' = dl',$$

$$\Sigma = aA_0' = al', \quad \Sigma' = a'A_0 = a'l.$$

We see that the statement is proved when we prove that every point S of Σ is a point S' of Σ'. This is true evidently for a point S which is a point A of a or a point A' of a' or, by the discussion of the third remark for $k = 2$, a point of the plane $D_n A_0 A_0' = D_n A_0' A_0 = ll' = l'l$.—We consider any other point S of Σ. Such a point S is collinear with certain points A of a and L' of l', where $L'AS$; and in its turn A, lying in $a = dl$ and lying neither in d nor in l, is collinear with certain points D of d and L of l, where LDA; and accordingly, by axiom 4, S is collinear with the point L and a point A' of the segment $L'D$ of the space $a = dl'$; this relation $LA'S$ indicates that the point S is a point S' of $\Sigma' = a'l$.—This completes the proof of statements 1 to 6 for the general k.

It remains to prove the validity of statement 7. We suppose that in the k-space Σ a p-space a and a q-space β $(p + q \geqq k)$ have a common point C_0. Then they have in common precisely an r-space γ, where $0 \leqq r \leqq p$, $0 \leqq r \leqq q$. We are to show that $r \geqq p + q - k$.

We take γ as determined by the $r + 1$ linearly independent points $C_0, \cdots, C_r$; and a as determined by these points with $p - r$ points $A_1, \cdots, A_{p-r}$; and β as determined by these points with $q - r$ points $B_1, \cdots, B_{q-r}$. If β lies in a, then $r = q$; hence, since $k \geqq p$, we have indeed $r \geqq p + q - k$.

We suppose then that β does not lie in a; thus there is a point B_1. Then the $(p + 1)$-space aB_1 has in common with the q-space β precisely the $(r + 1)$-space γB_1. For let B' be any point common to aB_1 and β. If B' is of a, then it is of γ and thus of γB_1. If it is not of a, then in the $(p + 1)$-space aB_1 it lies either on the opposite side of a from B_1, in which case the segment $B_1 B'$ contains a point A of a which is, however, likewise a point B, and thus is a point C of γ, and thus B' is of the line CB_1 and hence of the $(r + 1)$-space γB_1; or it is on the same side of a as B_1, in which case a similar result will follow by the mediation of a point B_1' lying on the line through B_1 and the point C_0 and on the opposite side of a from the point B_1. So we see that if the spaces a and β have γ in common, then the spaces aB_1 and β have precisely γB_1 in common. Hence aB_1 does not contain B_2. Similarly, aB_1B_2 and β have precisely $\gamma B_1 B_2$ in common, and aB_1B_2 does not contain B_3. Thus it appears finally that the points $C_0, \cdots, C_r, A_1, \cdots, A_{p-r}, B_1, \cdots, B_{q-r}$ are $1 + p + q - r$ independent points of the k-space Σ, and accordingly we have $p + q - r \leqq k$, that is, $r \geqq p + q - k$, which was to be proved.

§ 4. *The figure in a k-space determined by $k+1$ independent points.*

In a k-space a $(k+1)$-gon $A_1 \cdots A_{k+1}$ whose $k+1$ vertices A_g are independent points determines by its $k+1$ $(k-1)$-space faces $a_1 \cdots a_{k+1}$ a certain partition of the k-space.

To every point P of the k-space there is assigned a certain $(k+1)$-partite character; every partial character of P is $+$, 0, or $-$; the gth partial character is 0 if P lies in a_g, $\pm$ if P and A_g lie on $\begin{smallmatrix}\text{the same side}\\ \text{opposite sides}\end{smallmatrix}$ of a_g. Thus, for instance, A_1 has the character $(+, 0, \cdots 0)_{k\ 0's}$. From the gth partial characters of two points P_1, P_2 one infers at once whether the segment P_1P_2 has lying in a_g no point, one point, or all its points.

Points having the same character are classed together and form one of the parts in the partition in question. There are 3^{k+1} characters; of these, however, only those containing at least one partial character $+$ are characters of points. This is provable by induction. One notes, for instance, that the $(k+1)$-partite characters of points of the k-space $a_{k+1} = A_1 \cdots A_k$ are the k-partite characters of these points as points of a_{k+1} with the adjunction of 0 as $(k+1)$th character.

The points with character $(+, \cdots, +)_{k+1\ +'s}$ constitute the interior * of the $(k+1)$-gon, while similarly the points with no partial character $-$ constitute, as variously grouped, the interiors of the various faces of various dimensions of the $(k+1)$-gon. If P is a point of the interior of the $(k+1)$-gon, the line A_gP cuts the interior of the face a_g in a point A_g' such that P lies between A_g and A_g'.

We speak of complementary or opposite faces of the $(k+1)$-gon.† If a point P lies on no face and also not in the interior of the $(k+1)$-gon, then (in only one way) there are two points Q, R lying in the interior of opposite faces of the $(k+1)$-gon and such that Q lies on the segment PR. The character of P is made up of say q $+$'s and r $-$'s $(q+r=k+1;\ q>0, r>0)$, and, by convenient determination of the notation, it is $(+, \cdots, +, -, \cdots, -)_{q\ +'s;\ r\ -'s}$; then the points Q, R have respectively the characters

$$(+, \cdots, +, 0, \cdots, 0)_{q\ +'s;\ r\ 0's}, \quad (0, \cdots, 0, +, \cdots, +)_{q\ 0's;\ r\ +'s}.$$

The definition of the k-space analogous to the Peano-Ingrami definition ‡ of the plane and of the 3-space is the following:

A $(k+1)$-gon $A_1 \cdots A_{k+1}$ lying in no $(k-1)$-space determines as its interior the set of all points lying on the segments connecting its vertices to points lying in the interior of the respectively opposite k-gon faces; and it determines as the k-space $A_1 \cdots A_{k+1}$ the set of all points lying either on one of the $(k-1)$-space

* For $k=0$: a point constitutes simultaneously a 0-space, a 1-gon, the interior of a 1-gon.

† For example: $(k=2)$ in a triangle a vertex and opposite side are complementary faces; $(k=3)$ in a tetrahedron a vertex and opposite face are complementary faces, and two opposite edges are complementary faces.

‡ Cf. SCHUR, loc. cit., p. 268.

faces of the $(k+1)$-gon or on a line intersecting the interiors of two opposite faces of the $(k+1)$-gon.

This definition and that of § 1 are equivalent. Although in this definition the $k+1$ determining points enter symmetrically, the definition of § 1 is apparently much simpler.

§ 5. *The Hilbert plane.*

We consider a Hilbert geometry in which the basal elements are points, lines, segments of lines and Hilbert planes subject to the Hilbert axioms I 1, 2, 7_1, II 1, 2, 3, which do not involve the Hilbert plane, and furthermore subject to the following axioms I $7'_3$, $7'_2$, 3′, 5′, II 5′, which do involve and furnish the implicit definition of the Hilbert plane. These axioms I $7'_3$, $7'_2$, 3′, 5′, II 5′ are (by use of I 1, 2, 7_1, II 1, 2, 3) deducible from the Hilbert axioms I 7_3, 7_2, 3, 5, II 5.

I $7'_3$, $7'_2$. *There exists a Hilbert plane. Every Hilbert plane π has a notation ABC with respect to certain* three non-collinear points A, B, C. In the notation ABC there is no question of order of the three points A, B, C.*

I 3′. *Three non-collinear points A, B, C determine* a set of points: a Hilbert plane ABC, to which they belong.*

I 5′. *With respect to a Hilbert plane ABC every line joining any point A' of the segment BC to a point D distinct from A' and lying in the plane lies wholly in the plane.*

II 5′. (*The plane order axiom*). *With respect to a Hilbert plane ABC every line which lies in the plane and which contains a point of the segment BC contains also A or B or C or a point of the segment CA or a point of the segment AB.*

In this geometry we prove first the validity, as theorems, of the axioms 1–5 and 6 ($n=2$) of §1; from them follow by §2 the validity as a theorem of the Hilbert axiom II 4, by §§ 3, 4 the theory of the plane as defined in § 1, and the identity of this plane with that defined by PEANO.

Now we consider any Hilbert plane ABC with notation connected with three non-collinear points A, B, C. It contains the points A, B and C, the line BC (in view of the existence of a point A' of the segment BC), the lines CA and AB, and all lines of the types AA', BB', CC' where A', B', C' are points of the respective segments BC, CA, AB. Hence the Hilbert plane ABC contains the Peano plane ABC. Further, if D is a point of the Hilbert plane ABC not on one of the lines BC, CA, AB, then the line $A'D$ (where A' is a point of the segment BC) contains the point A or a point B' of the segment CA or a point C' of the segment AB. The line $A'D$ quâ $A'A$ or $A'B'$ or $A'C'$

* It is not implied that a plane π has such notation ABC in a definite way, nor that a notation ABC denotes a definite plane.

lies entirely in the plane ABC (of § 1). Hence this plane ABC contains the Hilbert plane ABC.

Accordingly the plane ABC (of § 1), the Peano plane ABC and the Hilbert plane ABC are identical. Then by § 3 follows the validity as theorems for the Hilbert plane of the Hilbert axioms I 3, 4, 5, 7_2, II 5 in their fulness.

Hence *the Hilbert axioms* I 4, II 4 *are logically deducible from the Hilbert axioms* I 1–3, 5, 7, II 1–3, 5.—If we had wished to exhibit merely this dependence it would have been simpler not to introduce the axioms I $7'_2$, $7'_3$, $3'$, $5'$, II $5'$.

THE UNIVERSITY OF CHICAGO,
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